MIT - Mechanical Engineering Department

2.12 Introduction to Robotics - Fall 2016 - Quiz 2

Date: November 21, 2016 | Duration: 80 minutes

Notes:

- There are **three** problems totaling 100 points.
- You have **80 minutes** to solve them.
- You are allowed to bring one two-sided handwritten letter-sized paper.
- Show the process to get to the answer, and draw a rectangle around the solution.

Let's Play Catch

The RRR robot manipulator in Figure 1 is tasked with catching a ball. In this exam, you will study how to control it. There are a total of 3 problems related to this system and they can be solved independently. Assume for all problems that:

- The length, mass, and angular inertia of link i is given correspondingly by l_i, m_i, and I_i.
- Link 1 and Link 2 have their center of mass exactly half way between their ends. The center of mass of Link 3 is concentric with the third rotational joint.
- The object is caught at point A, the center of the third rotational joint.

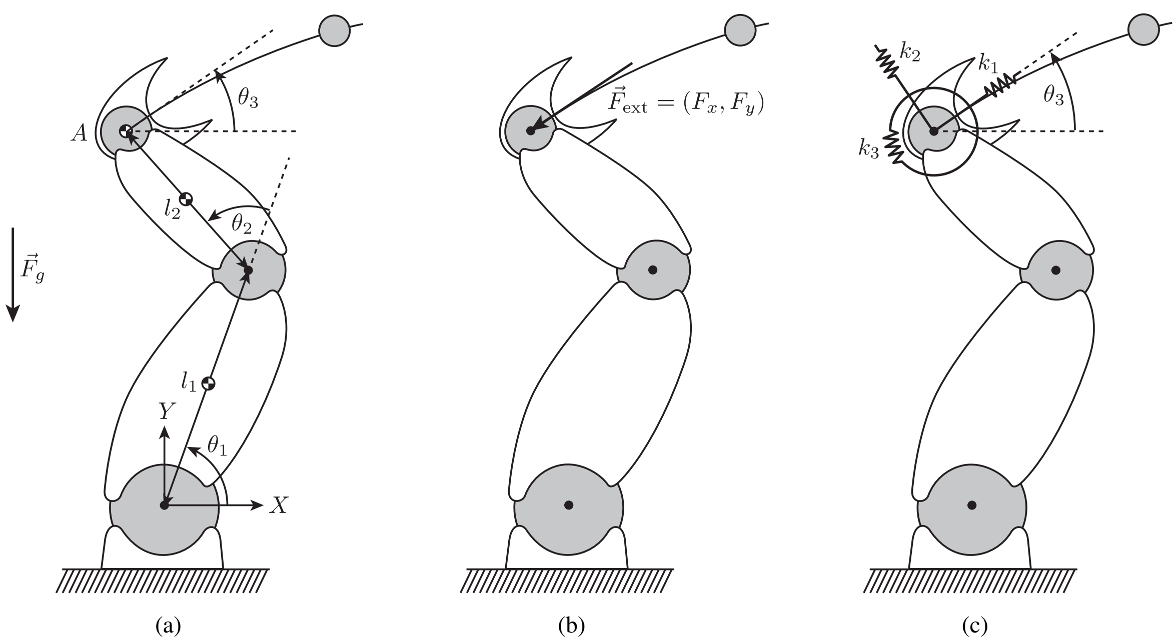

Figure 1: a) Geometry and generalized coordinates (θ_1, θ_2, θ_3). Note that angles θ_1 and θ_3 are defined absolute with respect to the horizontal, while θ_2 is defined relative to the orientation of link 1. b) Expected impact force while catching the ball. c) Desired cartesian stiffness (k_1, k_2, k_3) to soften the catch.

Problem 1 (30 points)

Gravity and Impact Force Cancellation

The robot arm is subject to gravity and the impact force of the ball. It will fall if no actuator torques are provided to compensate for them.

1. Using the principle of Virtual Work, compute the **torques** (τ_1, τ_2, τ_3) **to compensate gravity**, at the three rotational joints before the object comes into contact with the manipulator.

2. As depicted in Figure 1b, the object impacts the hand of the manipulator at point A with a force $\vec{F}_{\text{ext}} = (F_x, F_y)$. Compute the **torques** (τ_1, τ_2, τ_3) **to resist the impact**. Neglect at this point dynamic effects of the impact and inertia of the object.

Problem 2 (20 points)

Compliant Control

To prevent the ball from bouncing off from the hand of the manipulator, and to compensate for possible errors in the estimation of the trajectory of the ball, we decide to add programmed compliance at the end-effector (see Figure 1c). We want the robot to have low stiffness (k_1) along the direction of impact, and be more stiff along the other two directions (k_2 and k_3). Note that the direction of impact is directly given by the orientation of the last link of the robot.

1. Compute the **Joint Feedback Gain Matrix** K_q as a function of the joint angles $(\theta_1, \theta_2, \theta_3)$.

Problem 3 (50 points)

Dynamic Catching

Experiments show that reactive compliant control is not sufficient to catch the ball and avoid bouncing. For a better reception, we want the robot to also match the trajectory of the ball during the catch to minimize its impact. In order to accomplish this we need the governing dynamic equations of motion of the manipulator. For this problem assume that all masses are the same ($m = m_1 = m_2 = m_3$), all angular inertias are the same ($I = I_1 = I_2 = I_3$), and links 1 and 2 have the same length ($l = l_1 = l_2$).

1. Derive the expression of the **kinetic energy** for each link, as functions of generalized coordinates and their derivatives.

2. Derive the expression of the **potential energy** for each link, as functions of the generalized coordinates and their derivatives.

3. Compute the **generalized forces** (Q_1, Q_2, Q_3) induced by actuator torques (τ_1, τ_2, τ_3).

4. Derive the equations of motion using the **Lagrange method** prior to the impact of the object (i.e., no external forces other than actuator torques and gravity).

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Problem 1 (30 points)

Grading

1. 20 points
2. 10 points

Gravity and Impact Force Cancellation - Solution

Part 1:

First we write the position vectors of the center of masses of the links, then take variations with respect to the generalized coordinates:

$$r_1 = \begin{bmatrix} \frac{l_1}{2}c_1 \\ \frac{l_1}{2}s_1 \end{bmatrix}$$
$$r_2 = \begin{bmatrix} l_1c_1 + \frac{l_2}{2}c_{12} \\ l_1s_1 + \frac{l_2}{2}s_{12} \end{bmatrix}$$
$$r_3 = \begin{bmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \end{bmatrix}$$

Taking variations we have:

$$\delta r_1 = \begin{bmatrix} -\frac{l_1}{2}s_1\delta\theta_1 \\ \frac{l_1}{2}c_1\delta\theta_1 \end{bmatrix}$$
$$\delta r_2 = \begin{bmatrix} (-l_1s_1 - \frac{l_2}{2}s_{12})\delta\theta_1 + (-\frac{l_2}{2}s_{12})\delta\theta_2 \\ (l_1c_1 + \frac{l_2}{2}c_{12})\delta\theta_1 + (\frac{l_2}{2}c_{12})\delta\theta_2 \end{bmatrix}$$
$$\delta r_3 = \begin{bmatrix} (-l_1s_1 - l_2s_{12})\delta\theta_1 + (-l_2s_{12})\delta\theta_2 \\ (l_1c_1 + l_2c_{12})\delta\theta_1 + (l_2c_{12})\delta\theta_2 \end{bmatrix}$$

The gravitational forces are:

$$F_1 = \begin{bmatrix} 0 \\ -m_1g \end{bmatrix}$$
$$F_2 = \begin{bmatrix} 0 \\ -m_2g \end{bmatrix}$$
$$F_3 = \begin{bmatrix} 0 \\ -m_3g \end{bmatrix}$$

Next we compute:

$$\tau_{1,g}\delta\theta_1 + \tau_{2,g}\delta\theta_2 + \tau_{3,g}\delta\theta_3 = F_1 \cdot \delta r_1 + F_2 \cdot \delta r_2 + F_3 \cdot \delta r_3$$

Carrying out the computation:

$$\tau_{1,g} = -m_1 g \frac{l_1}{2} c_1 - m_2 g(l_1 c_1 + \frac{l_2}{2} c_{12}) - m_3 g(l_1 c_1 + l_2 c_{12})$$
$$\tau_{2,g} = -m_2 g(\frac{l_2}{2} c_{12}) - m_3 g(l_2 c_{12})$$
$$\tau_{3,g} = 0$$

Part 2:

In this section we study the manipulator in a static pose and use the relation:

$$\tau = -J^T F_{impact}$$

The reason for the negative sign is that the manipulator must resist the impact force so it must apply a force equal and opposite in direction.

We have derived the Jacobian we need for this part in the previous section:

$$\delta r_3 = J \delta \theta = \begin{bmatrix} (-l_1 s_1 - l_2 s_{12}) & (-l_2 s_{12}) & 0 \\ (l_1 c_1 + l_2 c_{12}) & (l_2 c_{12}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \delta\theta$$

and:

$$F_{impact} = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}$$

Problem 2 (20 points)

Grading

1. 20 points

Compliant Control of Catching RRR Manipulator - Solution

The compliance along the axis and perpendicular to it:

$$F_a = K_p \delta a = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \delta a$$

where δa is the displacement in the frame of impact where the first direction is axial to link 3.

Now we apply a rotation to get to xy coordinate frame orientation:

$$F_{xy} = R K_p R^T \delta p$$

where:

$$R = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next use Jacobian to get to the joint feedback gain matrix:

$$\tau = J^T R K_p R^T J \delta q$$
$$K_q = J^T R K_p R^T J$$

where:

$$J = \begin{bmatrix} (-l_1 s_1 - l_2 s_{12}) & (-l_2 s_{12}) & 0 \\ (l_1 c_1 + l_2 c_{12}) & (l_2 c_{12}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3 (50 points)

Grading

1. 15 points
2. 10 points
3. 10 points
4. 15 points

RRR Manipulator Dynamics - Solution

Part 1

Derivation with $m_1 = m_2 = m_3 = m$, $l_1 = l_2 = l$, $I_1 = I_2 = I_3 = I$:
The velocities of the centers of mass are given by:

$$v_{c,1} = \frac{l}{2}(-\sin\theta_1 \hat{e}_x + \cos\theta_1 \hat{e}_y)$$
$$v_{c,2} = l((-\sin\theta_1 - 0.5\sin(\theta_1+\theta_2)\hat{e}_x + (\cos\theta_1 + 0.5\cos(\theta_1+\theta_2))\hat{e}_y)$$
$$v_{c,3} = l((-\sin\theta_1 - \sin(\theta_1+\theta_2)\hat{e}_x + (\cos\theta_1 + \cos(\theta_1+\theta_2))\hat{e}_y)$$

Then we may write:

$$T = \sum_{i=1}^{3} T_i = \sum_{i=1}^{3} \frac{1}{2}m(v_{c,i}^T v_{c,i}) + \frac{1}{2}I\omega_i^2$$

where:

$$T_1 = \frac{1}{2}ml^2(\frac{1}{4}\dot{\theta}_1^2) + \frac{1}{2}I\dot{\theta}_1^2$$
$$T_2 = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \frac{1}{4}(\dot{\theta}_1+\dot{\theta}_2)^2 + (\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\cos\theta_2) + \frac{1}{2}I(\dot{\theta}_1+\dot{\theta}_2)^2$$
$$T_3 = \frac{1}{2}ml^2(\dot{\theta}_1^2 + (\dot{\theta}_1+\dot{\theta}_2)^2 + 2(\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_1\cos\theta_2) + \frac{1}{2}I\dot{\theta}_3^2$$

Part 2

$$U_1 = mgl(\frac{1}{2}\sin\theta_1)$$
$$U_2 = mgl(\sin\theta_1 + \frac{1}{2}\sin(\theta_1+\theta_2))$$
$$U_3 = mgl(\sin\theta_1 + \sin(\theta_1+\theta_2))$$

Part 3

To find a relationship between the generalized forces and torques, lets define $\theta_3^{'} = \theta_3 - (\theta_1+\theta_2)$, now with this definition the position of link 3 described by $\theta_3^{'}$ is defined relative to link two and so we know for this case that:

$$\sum Q_i \delta\theta_i = Q_1\delta\theta_1 + Q_2\delta\theta_1 + Q_3\delta\theta_3^{'}$$
$$Q_1\delta\theta_1 = \tau_1\delta\theta_1$$
$$Q_2\delta\theta_2 = \tau_2\delta\theta_2$$
$$Q_3\delta\theta_3^{'} = \tau_3\delta\theta_3^{'} = \tau_3(\delta\theta_3 - (\delta\theta_1 + \delta\theta_2))$$

Re-arranging the above we have:

$$Q_1 = (\tau_1 - \tau_3)$$
$$Q_2 = (\tau_2 - \tau_3)$$
$$Q_3 = \tau_3$$

Part 4

Next we use the Lagrangian method to derive the dynamic equations of motion:

$$\frac{\partial T}{\partial \dot{\theta}_1} = (ml^2(\frac{7}{2} + 3\cos\theta_2) + 2I)\dot{\theta}_1 + (ml^2(\frac{5}{4} + \frac{3}{2}\cos\theta_2) + I)\dot{\theta}_2$$
$$\frac{\partial T}{\partial \dot{\theta}_2} = (ml^2(\frac{5}{4} + \frac{3}{2}\cos\theta_2) + I)\dot{\theta}_1 + (\frac{5}{4}ml^2 + I)\dot{\theta}_2$$
$$\frac{\partial T}{\partial \dot{\theta}_3} = I\dot{\theta}_3$$
$$\frac{\partial T}{\partial \theta_1} = 0$$
$$\frac{\partial T}{\partial \theta_2} = -\frac{3}{2}ml^2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2$$
$$\frac{\partial T}{\partial \theta_3} = 0$$
$$\frac{\partial U}{\partial \theta_1} = mgl(\frac{5}{2}\cos\theta_1 + \frac{3}{2}\cos(\theta_1 + \theta_2))$$
$$\frac{\partial U}{\partial \theta_2} = mgl(\frac{3}{2}\cos(\theta_1 + \theta_2))$$
$$\frac{\partial U}{\partial \theta_3} = 0$$

Using:

$$L = T - V$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau$$

We may write:

$$\begin{bmatrix} a_1 + 2a_2\cos\theta_2 & a_2\cos\theta_2 + a_3 & 0 \\ a_2\cos\theta_2 + a_3 & a_3 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} -2a_2\sin\theta_2\dot{\theta}_2 & -a_2\sin\theta_2\dot{\theta}_2 & 0 \\ a_2\sin\theta_2\dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta} + \frac{\partial U}{\partial \theta} = \begin{bmatrix} \tau_1 - \tau_3 \\ \tau_2 - \tau_3 \\ \tau_3 \end{bmatrix}$$

with:

$$a_1 = \frac{7}{2}ml^2 + 2I$$
$$a_2 = \frac{3}{2}ml^2 \cos\theta_2$$
$$a_3 = \frac{5}{4}ml^2 + I$$