→ Algebraic definition

\[
\text{det}\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

\[
\text{det}\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{43} + a_{31}a_{42}a_{53} - a_{11}a_{23}a_{32} - a_{21}a_{33}a_{42} - a_{31}a_{43}a_{52}
\]

\[
\text{det}\begin{bmatrix}
a_{11} & \cdots & a_{1M} \\
\vdots & \ddots & \vdots \\
a_{N1} & \cdots & a_{NN}
\end{bmatrix} = \sum a_{ni} (-1)^{i+n} \text{det} (A_{ni})
\]

\[\text{Matrix resulting of eliminating column }i \text{ and row }1.\]

Example:

\[
\text{det}\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} = a_{11} \text{det} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12} \text{det} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \text{det} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\]

→ Geometric Interpretation:

In 2D:

\[
\text{det}\begin{bmatrix} \vec{v}_1 \vec{v}_2 \end{bmatrix} = \text{AREA parallelogram with vectors } \vec{v}_1 \text{ and } \vec{v}_2 \text{ as sides.}
\]

\[\text{NOTE: If } \text{det} = 0 \Rightarrow \text{Area} = 0\]

Both vectors are aligned.
In 3D:

\[
\begin{vmatrix}
\vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\
\end{vmatrix} = \text{VOLUME parallelepiped with } \vec{v}_1, \vec{v}_2 \text{ and } \vec{v}_3 \text{ as edges.}
\]

**NOTE:** This implies that if \( \text{det} = 0 \) then volume is zero, which means that the three vectors are in a plane, so:
- They are linearly dependent.
- They do not span all 3D space.

→ Properties:

- \( \text{det} (A^T) = \text{det} (A) \)
- \( \text{det} (A^{-1}) = \frac{1}{\text{det} (A)} \)
- \( \text{det} (A \cdot B) = \text{det} (A) \cdot \text{det} (B) \)
- \( \text{det} (R) = +1 \) if \( R \) is a rotation matrix.
- \( \text{det} \left[ \vec{v}_1 \ldots k \cdot \vec{v}_i \ldots \vec{v}_n \right] = k \cdot \text{det} \left[ \vec{v}_1 \ldots \vec{v}_i \ldots \vec{v}_n \right] \)
- \( \text{det} (k \cdot A) = k^n \cdot \text{det} (A) \) with \( n \) the dimension of \( A \).