# HW 1 Solutions:

## Problem 1:

a) Recall that the governing equation is

$$\tau_m = \frac{k_t}{R} u - \frac{k_t^2}{R} \omega_m$$

Here $u = 24$ volts, to find $k_t$, R first plug in:

① $\tau = 0$ N-m, $\omega_m = 200$ rad/s $\Rightarrow$ $\boxed{k_t = 0.12}$ $\frac{V.S}{rad}$ or $\frac{N.m}{S}$

- torque constant
- back emf constant

② $\tau = 4\times10^{-3}$ N-m, $\omega_m = 0$ rad/s $\Rightarrow$ $\boxed{R = 720\ \Omega}$

Recall from class that to find the motor constant from torque constant:

$$k_m = \frac{k_t}{\sqrt{R}} = 4.5\times10^{-3} \frac{N\text{-}m}{\sqrt{W}} \leftarrow \text{watt}$$

b) Back emf is given by $E = \underset{0.12}{k_t}\, \underset{100}{\omega_m} \rightarrow E = 12$ volt

c) Note for power dissipation $\rightarrow$ $\begin{cases} P = Ri^2 \\ i = \frac{\tau_m}{k_t^2} \end{cases} \Rightarrow P = R\frac{\tau_m^2}{k_t^2}$

Further $\rightarrow$ $\boxed{P = \frac{\tau_m^2}{k_m^2}}$ $\rightarrow$ $P = \frac{(2\times10^{-3})^2}{(4.5\times10^{-3})^2} = 0.196$ Watt $\simeq 0.2$ Watt

$k_m = \frac{k_t}{\sqrt{R}}$

d) Recall that in class we derived

$$I\dot{\omega} + b\omega = \frac{rk_t}{R}u$$

which relates the angular velocity to the voltage input.

Here $\omega \simeq 0$ and $I \equiv I_a + r^2 I_m$, plug in and solve for $\dot{\omega}$:

$$\dot{\omega} = \frac{k_t}{R}\frac{r}{I_a + r^2 I_m}u$$

we want to maximize this expression so we differentiate it with respect to $r$ and set to zero:

$$\frac{\partial}{\partial r} \Rightarrow \left(I_a + r^2 I_m - r(2I_m r) = 0\right) \Rightarrow \boxed{r_{opt} = \sqrt{\frac{I_a}{I_m}}}$$

plug in $\Rightarrow$ $\boxed{r_{opt} = 40}$

e) $\dot{\omega}_{max} = \frac{k_t}{R}\frac{r_{opt}}{I_a + r_{opt}^2 I_m}u = \frac{0.12}{720}\frac{40}{1.6\times10^{-1} + 40^2 \times 1\times10^{-4}} \times 30$

$\dot{\omega}_{max} = 0.625 \ \text{rad}/s^2$

f) $E = \underbrace{k_{emf}}_{= k_t}\omega = k_t\omega = 5\times10^{-2}\times100 = \boxed{5V}$

g) We will first find how much heat is generated in 1 cycle then multiply it by the number of cycles:

$$W_{heat} = \int P_{dis} \cdot dt$$

$$P_{dis} = V_{ce} \cdot I_c$$

to be able to integrate we need to write both $V_{ce}$ & $I_C$ as functions of time:

$$\begin{cases} I_C = 0 & \text{at } t=0 \\ I_C = 0.5 & \text{at } t = 5\mu s \end{cases} \rightarrow I_C = \frac{0.5A}{5\mu s} t$$

$$\boxed{I_C = 0.1 \times 10^6 t} \leftarrow \text{For ON phase}$$

$$\begin{cases} V = 10 & \text{at } t=0 \\ V = 0 & \text{at } t = 5\mu s \end{cases} \rightarrow V_{CE} = 10 - \frac{10V}{5\mu s} t$$

$$\boxed{V_{CE} = -2 \times 10^6 t + 10}$$

$$\hookrightarrow P_{dis} = I_C \cdot V_{CE} \Rightarrow W_{ON} = \int_0^{T_{ON}} P_{dis}\, dt = \frac{125}{3} \times 10^{-7} J$$

Note that:

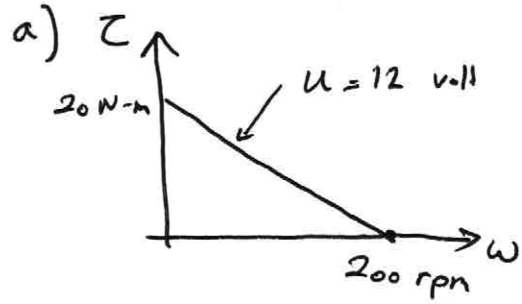
$$W_{OFF} = 2\, W_{ON} = \frac{2}{3}\, 125 \times 10^{-7} J$$

$$\hookrightarrow W_{1\ cycle} = W_{ON} + W_{OFF} = 1.25 \times 10^{-5}\ J/cycle$$

$$W_{1\ sec} = W_{1\ cycle} \times \text{Number of cycles} = 1.25 \times 10^{-1}\ J/s \equiv 3 \times 10^{-2}\ Cal/s$$

## Problem 2:

a)




$$\tau_m = \frac{k_t}{R} u - \frac{k_t^2}{R} \omega_m$$

$(\tau = 0,\ \omega = 200\text{rpm})$

$$\frac{u}{R} = \frac{\omega_m}{R} k_t \rightarrow u = \omega_m k_t$$

$$\boxed{k_t = \frac{12 \text{ volt}}{200 \text{ rpm}} \simeq \frac{12}{21 \text{ rad/s}} = 0.57}$$

$$(\tau = 20,\ \omega = 0) \rightarrow 20 = \frac{0.57}{R} \times 12 \rightarrow \boxed{R = 0.342}$$

We know the maximum power is when the motor is running at half its max torque output

$$\left\{ \begin{array}{l} \tau_{mid} = 10 \text{ N-m} \\ \omega_{mid} = 10.53 \text{ rad/sec} \end{array} \right. \rightarrow \boxed{P_{max} = \tau_{mid} \cdot \omega_{mid} = 105.3 \text{ watt}}$$

b) The problem is equivalent to finding $n$ s.t.:

$$\frac{360}{2^n} < 0.1 \rightarrow n > 12 \rightarrow n_{min} = 12 \rightarrow 2^n = 4096$$

which requires 12 bits of information to represent.

c) Let's assume motor runs at maximum speed $\rightarrow \omega = 200\text{rpm}$

and there are 4096 ticks ...

... this means we recieve $p = \frac{200 \text{ rpm} \times 4096}{60 \text{ sec}}$

$\leadsto$ $p = 13653.33$ pulses/second (changes in the optical sensor reading out put) So we need a chip operating at least as fast $\rightarrow$ Sampling rate $> 13653.33$

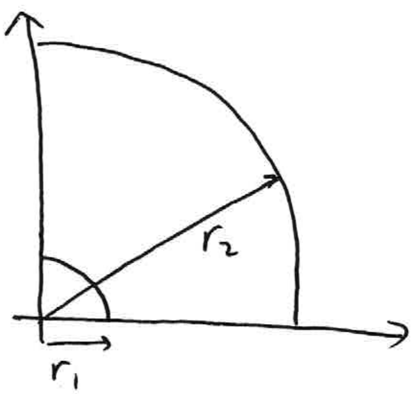
Problem 4:

a) From the geometry of the manipulator we write:

$$x = l_1 \cos q_1 + q_2 \cos q_1 + l_3 \cos(q_1 + q_3)$$
$$y = l_1 \sin q_1 + q_2 \sin q_1 + l_3 \sin(q_1 + q_3)$$

This mapping is 1 to 1 because the output of the sine and cosine functions is unique for a given $\vec{q}$

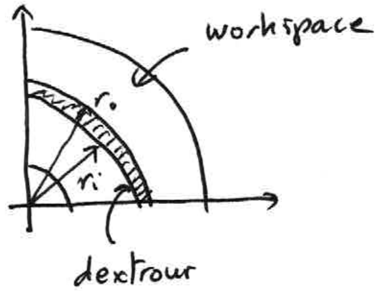
b)




$r_1 = l_2 - l_3$
$r_2 = l_1 + l_2 + l_3$
repeat for other 3 quadrants.

The dextrous work space will depend on the relative lengths of the links, answers reflecting this fact will get full credit.

If $l_3 < l_2$:




$r_i = l_1 + l_3$
$r_o = l_1 + l_2 - l_3$

If $l_3 > l_2$:

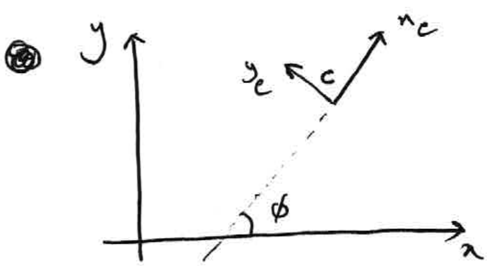
The workspace that we consider dextrous is null in this case, it does not exist.

★ $(x,y) \in$ Dextrous workspace if $\phi \in [0, 2\pi]$ So for any given $(x,y)$ $\phi$ can take any value between $[0, 2\pi]$ without changing $(x,y)$ and satisfying constraints on $q_i$

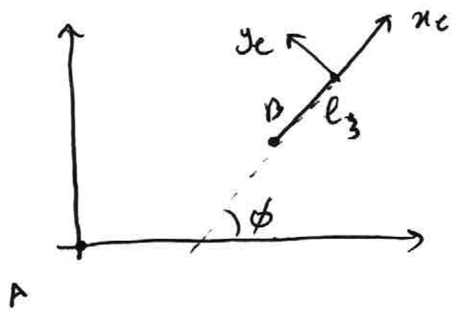
c) The inverse map from $(x, y, \phi) \rightarrow (q_1, q_2, q_3)$ is unique but the inverse map from $(x, y) \rightarrow (q_1, q_2, q_3)$ is not unique.

Geometric demonstration:

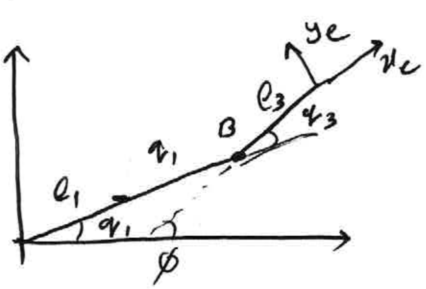
① Consider the reference frame of the end effector attached at pt C:




② Draw line segment representing the $3^{rd}$ link from C with angle $\phi$ from the horizontal

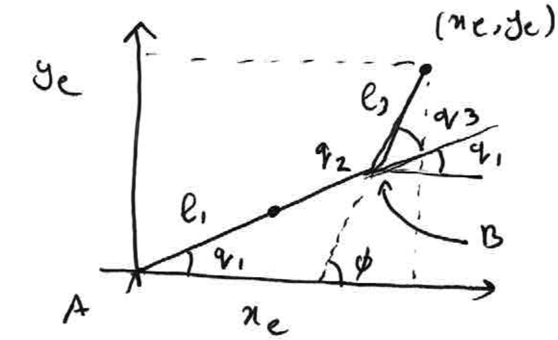



③ Draw a line connecting A to B, and denote the first segment as $l_1$ and the remaining as $q_2$




④ This is geometrically the only way to arrive at this configuration so the map is unique.

Algebraic demonstration




From the figure we can write:

$$\phi = q_1 + q_3$$

$$\begin{pmatrix} x_B \\ y_B \end{pmatrix} = \begin{pmatrix} x_e - l_3 \cos\phi \\ y_e - l_3 \sin\phi \end{pmatrix}$$

$$\tan q_1 = \frac{y_B}{x_B} \quad \rightsquigarrow$$

$$\boxed{\begin{aligned} q_1 &= \operatorname{atan}\frac{y_B}{x_B} = \operatorname{atan}\frac{(y_e - l_3\sin\phi)}{(x_e - l_3\cos\phi)} \\ q_3 &= \phi - \operatorname{atan}\frac{(y_e - l_3\sin\phi)}{(x_e - l_3\cos\phi)} \\ q_2 &= \sqrt{(x_e - l_3\cos\phi)^2 + (y_e - l_3\sin\phi)^2} - l_1 \end{aligned}}$$

We apply the constraints that $q_2 \in [0, l_2]$ and $0 < l_2 < l_1$ then the ~~atan~~ atan functions only give one acceptable answer.

d) This manipulator has 3 d.o.f. allowing us to place the manipulator end-effector at desired pose $(x_e, y_e)$ and specify the orientation, but we cannot do this with the 2 link manipulator.

e) See part 2 of solution to c.