Massachusetts Institute of Technology
Department of Mechanical Engineering
2.12 – Introduction to Robotics

Problem Set 4
Solutions

PROBLEM 1 (20 pts)

Part (a)

The forward kinematic equations relating the end-effector position and orientation to the joint displacements may be obtained by inspection of the figure given in the problem statement. The equations are

$$\begin{aligned} x_e &= \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2 - \pi) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3 - 2\pi) \\ y_e &= \ell_1 \sin\theta_1 - \ell_2 \sin(\theta_1 + \theta_2 - \pi) - \ell_3 \sin(\theta_1 + \theta_2 + \theta_3 - 2\pi) \\ \phi_e &= \theta_1 + \theta_2 + \theta_3 - 2\pi \end{aligned} \tag{1.1}$$

Equivalently,

$$\begin{aligned} x_e &= l_1 c_1 - l_2 c_{12} + l_3 c_{123} \\ y_e &= l_1 s_1 - l_2 s_{12} + l_3 s_{123} \\ \phi_e &= \theta_1 + \theta_2 + \theta_3 - 2\pi \end{aligned} \tag{1.2}$$

Taking partial derivatives we can construct the Jacobian:

$$\mathbf{J} = \begin{pmatrix} -l_1 s_1 + l_2 s_{12} - l_3 s_{123} & l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 - l_2 c_{12} + l_3 c_{123} & -l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{pmatrix} \tag{1.3}$$

Evaluation of this matrix at $l_1 = 3m$, $l_2 = 2m$, $l_3 = 1m$, $\theta_1 = 135°$, $\theta_2 = 45°$, and $\theta_3 = 225°$ yields

$$\mathbf{J} = \begin{pmatrix} -2\sqrt{2} & \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ 2 - \sqrt{2} & 2 + \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 1 & 1 \end{pmatrix} \tag{1.4}$$

Part (b)

Based on the principle of virtual work (in the absence of friction and gravitational torques), we can compute the equivalent joint torques from $\mathbf{J}^T\mathbf{F}$:

$$
\begin{aligned}
\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \mathbf{J}^T\mathbf{F} &= \begin{pmatrix} -2\sqrt{2} & 2-\sqrt{2} & 1 \\ \dfrac{-\sqrt{2}}{2} & 2+\dfrac{\sqrt{2}}{2} & 1 \\ \dfrac{-\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} & 1 \end{pmatrix} \begin{pmatrix} 10N \\ -2N \\ 0.2N{\cdot}m \end{pmatrix} \\
&= \begin{pmatrix} -20\sqrt{2} - 4 + 2\sqrt{2} + \dfrac{1}{5} \\ -6\sqrt{2} - 4 + \dfrac{1}{5} \\ -6\sqrt{2} + \dfrac{1}{5} \end{pmatrix} N{\cdot}m \\
&= \begin{pmatrix} -29.26 \\ -12.29 \\ -8.29 \end{pmatrix} N{\cdot}m
\end{aligned} \tag{1.5}
$$

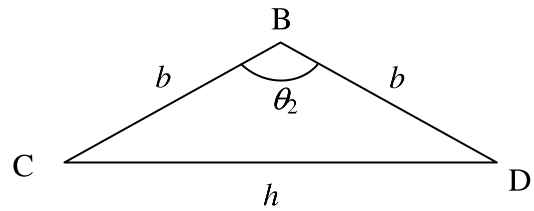
PROBLEM 4 (30 pts)

Part (a)

We will use θ_1 and θ_2 as our generalized coordinates. These angles completely specify the configuration of the system.

Part (b)

The length of the spring may be obtained from the isosceles triangle BCD:



Using the triangle we find

$$h = 2b\sin\left(\frac{\theta_2}{2}\right) \tag{4.1}$$

And therefore,

$$\begin{aligned}\delta h &= \frac{\partial h}{\partial \theta_2}\delta\theta_2\\ &=2b\cos\left(\frac{\theta_2}{2}\right)\delta\theta_2\end{aligned} \tag{4.2}$$

Part (c)

Here we will apply the principle of virtual work. The expression for the virtual work in terms of convenient coordinates is

$$\delta W = f\cdot\delta y_A + k(h-h_0)\cdot\delta h + F_B\cdot\delta y_B + F_E\delta y_E \tag{4.3}$$

Our task now is to rewrite the virtual work as a function of only generalized coordinates and their virtual displacements. We proceed as follows:

$$y_A = -l_0\sin\theta_1 \rightarrow \delta y_A = -l_0\cos\theta_1\delta\theta_1 \tag{4.4}$$

$$y_B = l_1\sin\theta_1 \rightarrow \delta y_B = l_1\cos\theta_1\delta\theta_1 \tag{4.5}$$

$$\begin{aligned}y_E &= y_B - l_2\sin(\pi-(\theta_2+\theta_1))\\ &= y_B - l_2\sin(\theta_1+\theta_2)\end{aligned} \tag{4.6}$$

In Eq. (4.6), we recognize that there is a term involving both θ_1 and θ_2. In general, for a multivariable term $g(\theta_1,\theta_2)$ where g is an arbitrary (but sufficiently smooth) function, we may write its variation as follows

$$\delta g = \frac{\partial g}{\partial\theta_1}\delta\theta_1 + \frac{\partial g}{\partial\theta_2}\delta\theta_2 \tag{4.7}$$

Applying this to Eq. (4.6) we find,

$$\delta y_E = \delta y_B - l_2 \cos(\theta_1 + \theta_2)(\delta\theta_1 + \delta\theta_2) \quad (4.8)$$

We now have all that is necessary to rewrite Eq. (4.3). We use Eqs. (4.1), (4.2) from part (b) and Eqs. (4.4), (4.5), (4.6), and (4.8) from our work above. This substitution yields,

$$\begin{aligned}\delta W = {} & f(-l_0 \cos\theta_1)\delta\theta_1 - 2bk\left(2b\sin\left(\frac{\theta_2}{2}\right) - h_0\right)\cos\left(\frac{\theta_2}{2}\right)\delta\theta_2 + F_b(l_1\cos\theta_1)\delta\theta_1 \\ & + F_E(l_1\cos\theta_1\delta\theta_1 - l_2\cos(\theta_1+\theta_2)(\delta\theta_1+\delta\theta_2))\end{aligned} \quad (4.9)$$

Rewriting as an inner product, we have

$$\delta W = \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix} \quad (4.10)$$

where

$$w_1 = -fl_0\cos\theta_1 + F_b l_1 \cos\theta_1 + F_E l_1 \cos\theta_1 - F_E l_2 \cos(\theta_1 + \theta_2) \quad (4.11)$$

$$w_2 = -2bk\left(2b\sin\left(\frac{\theta_2}{2}\right) - h_0\right)\cos\left(\frac{\theta_2}{2}\right) - F_E l_2\cos(\theta_1+\theta_2) \quad (4.12)$$

The virtual work must vanish for an arbitrary combination of $\delta\theta_1$ and $\delta\theta_2$, which implies that

$w_1 = 0$ and $w_2 = 0$.

For this part of the problem we are interested in the force *f*. We solve for *f* from the first equation. We find,

$$\begin{aligned} f &= \frac{F_b l_1\cos\theta_1 + F_E l_1 \cos\theta_1 - F_E l_2\cos(\theta_1+\theta_2)}{l_0\cos\theta_1} \\ &= F_b\left(\frac{l_1}{l_0}\right) + F_E\left(\frac{l_1}{l_0} - \frac{l_2}{l_0}\frac{\cos(\theta_1+\theta_2)}{\cos(\theta_1)}\right)\end{aligned} \quad (4.13)$$

Part (d)

The problem should read:

d). Obtain the joint angle θ_2 when the system with given forces F_b , F_e, and the actuator force *f* is in equilibrium. For simplicity set the parameter h_0 to zero; and assume $\theta_1 = 0$.

The equilibrium joint angle θ_2 may be obtained from Eq. (4.12) set to zero. Using $h_0 = 0$, $\theta_1 = 0$, and $2\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_2}{2}\right) = \sin(\theta_2)$,we have

$$\begin{aligned} & -2b^2k\sin(\theta_2) - F_E l_2\cos(\theta_2) = 0 \\ & i.e. \\ & \theta_2 = \tan^{-1}\left(-\frac{F_E l_2}{2b^2k}\right)\end{aligned} \quad (4.14)$$

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Problem Set 5
Solutions

PROBLEM 1 (15 pts)

	Kinematic	Static
Natural Constraints	$v_z = 0$ $\omega_x = 0$	$f_x = 0$ $f_y = 0$ $\tau_y = 0$ $\tau_z = 0$
Artificial Constraints	$v_x = R\omega_y$ $v_y = 0$ $\omega_y = +\Omega$ $\omega_z = 0$	$f_z = -F$ $\tau_x = 0$

The x-axis linear velocity and the y-axis angular velocity must be coordinated in order to have the roller roll on the xy plane without slip. Thus, $v_x = R\omega_y$. note that this no-slip condition is required to meet the task goal, although sliding is geometrically admissible. Therefore, this is an artificial constraint that must be satisfied through feedback control.