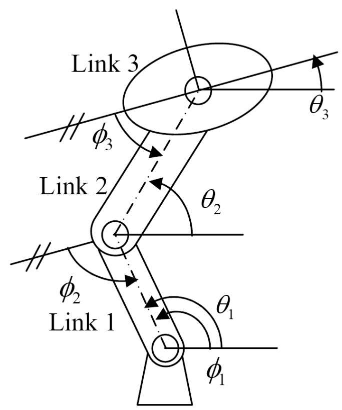
PROBLEM 3 (20 pts)

As discussed in the previous problem set, the relationship between actuator displacement and joint angles is given by

$$\begin{aligned} \phi_1 &= \theta_1 \\ \phi_2 &= \theta_1 - \theta_3 \\ \phi_3 &= \theta_2 - \theta_3 \end{aligned} \tag{3.1}$$

Recall that these angles can be determined by thinking of the actuator reference datums as shown in the figure below:



Differentiating these equations yields

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}, \quad \text{where } \mathbf{J}_{act} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \tag{3.2}$$

From the duality principle,

$$\mathbf{Q} = \mathbf{J}^T \mathbf{F} \quad \text{and} \quad \Delta \mathbf{p} = \mathbf{J} \Delta \theta \tag{3.3}$$

where $\mathbf{F} = [F_x \;\; F_y \;\; M]^T$ and $\mathbf{Q}$ is the vector of generalized forces corresponding to θ_1, θ_2, and θ_3. We collect the small joint displacement in the vector = $[\Delta\theta_1 \;\; \Delta\theta_2 \;\; \Delta\theta_3]^T$. Also, $\Delta\mathbf{p} = [\Delta x_h \;\; \Delta y_h \;\; \Delta\alpha]^T$. The desired hip stiffness is given by

$$\mathbf{K}_{hip} = \begin{pmatrix} k_x & & 0 \\ & k_y & \\ 0 & & k_\alpha \end{pmatrix}. \tag{3.4}$$

We can write $\mathbf{Q}$ as

$$\begin{aligned} \mathbf{Q} &= \mathbf{J}^T \mathbf{F} \\ &= \mathbf{J}^T \mathbf{K}_{hip} \Delta \mathbf{p} \\ &= \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \Delta \theta \end{aligned} \tag{3.5}$$

The feedback gain that is required is then given by

$$\mathbf{K} = \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \tag{3.6}$$

From the figure given in the problem, the Jacobian **J** is computed as

$$\begin{aligned}
\mathbf{J} &= \begin{pmatrix} -\sin\left(\frac{\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) & 0 \\ \cos\left(\frac{\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{3.7}$$

Therefore, the joint stiffness in terms of θ_1, θ_2, and θ_3 is given by

$$\boxed{\begin{aligned}
\mathbf{K} &= \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \\
&= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_x & & 0 \\ & k_y & \\ 0 & & k_\alpha \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(k_x + k_y) & \frac{1}{2}(k_x - k_y) & 0 \\ \frac{1}{2}(k_x - k_y) & \frac{1}{2}(k_x + k_y) & 0 \\ 0 & 0 & k_\alpha \end{pmatrix}
\end{aligned}} \tag{3.8}$$

To obtain the stiffness matrix in terms of ϕ_1, ϕ_2, and ϕ_3, consider the Jacobian $\mathbf{J}_{act}$:

$$\Delta\theta = \mathbf{J}_{act}{}^{-1} \Delta\phi \tag{3.9}$$

Also,

$$\tau = \mathbf{J}_{act}{}^{-T} \mathbf{Q} \tag{3.10}$$

We seek a matrix relating the actuator torques $\tau = [\tau_1 \quad \tau_2 \quad \tau_3]^T$ and the actuator displacements $\Delta\phi = [\Delta\phi_1 \quad \Delta\phi_2 \quad \Delta\phi_3]^T$:

$$\tau = \mathbf{K}_{act} \Delta\phi \tag{3.11}$$

We find

$$\begin{aligned}
\tau &= \mathbf{J}_{act}{}^{-T} \mathbf{Q} \\
&= \mathbf{J}_{act}{}^{-T} \mathbf{J}^T \mathbf{F} \\
&= \mathbf{J}_{act}{}^{-T} \mathbf{J}^T \mathbf{K}_{hip} \Delta\mathbf{p} \\
&= \mathbf{J}_{act}{}^{-T} \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \Delta\theta \\
&= \mathbf{J}_{act}{}^{-T} \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \mathbf{J}_{act}{}^{-1} \Delta\phi
\end{aligned} \tag{3.12}$$

From the last line of the above equation,

$$\boxed{\mathbf{K}_{act} = \mathbf{J}_{act}{}^{-T} \mathbf{J}^T \mathbf{K}_{hip} \mathbf{J} \mathbf{J}_{act}{}^{-1}} \tag{3.13}$$

1. Problem 2

In order to solve this problem we write force balance for each of the masses and compute the total sum of forces applied. Once we have expressions for each of the masses we will relate V to y_s through a state-space representation. This is not the only valid solution, if a Laplace domain relation is provided that is also great.

$$m_1\ddot{y}_1 = -k_1 y_1 - k_2(y_1 - y_2) - c_1\dot{y}_1 - c_2(\dot{y}_1 - \dot{y}_2)$$
$$m_2\ddot{y}_2 = k_2(y_1 - y_2) + c_2(\dot{y}_1 - \dot{y}_2) + k_3(y_s - y_2)$$

In the above we have assumed $y_s > y_2 > y_1$ though this assumption is not necessary, it is made so that the internal forces are written consistently, as long as you are consistent in that regard the solution will be correct. Re-arranging we have:

$$x = \begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \rightarrow \dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2+k_3}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_3}{m_2} \end{bmatrix} y_s$$

In this instance we can measure the voltage across the piezo-electric patch which is directly proportional to the displacement of the patch, we may write:

$$V = \begin{bmatrix} k & 0 & 0 & 0 \end{bmatrix} x$$

The above is the state space representation relating the sample height to the piezo-electric patch voltage measurement. You may provide solutions that are in differential form like the above or in Laplace domain where you have a relationship strictly between the voltage and the sample displacement.

2. Problem 3

We need to compute the kinetic and potential energies for the manipulator first. In order to compute the kinetic energy we must first evaluate the velocities of the centers of mass:

$$r_{c,1} = \begin{bmatrix} r_1 c_1 \\ r_1 s_1 \end{bmatrix} \rightarrow v_{c,1} = \begin{bmatrix} -r_1 s_1 \dot{\theta}_1 \\ r_1 c_1 \dot{\theta}_1 \end{bmatrix}$$

$$r_{c,2} = \begin{bmatrix} l_1 c_1 + r_2 c_{12} \\ l_1 s_1 + r_2 s_{12} \end{bmatrix} \rightarrow v_{c,1} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - r_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + r_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

Kinetic Energy:

$$T = \frac{1}{2} m_1 \left(v_{c,1}^T v_{c,1} \right) + \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(v_{c,2}^T v_{c,2} \right) + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

Replacing the values found previously for the velocities of the center of masses we have:

$$T = \frac{1}{2} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \begin{bmatrix} I_1 + I_2 + m_1 r_1^2 + m_2(l_1^2 + r_2^2) + 2 m_2 l_1 r_2 c_2 & I_2 + m_2 r_2^2 + m_2 l_1 r_2 \\ I_2 + m_2 r_2^2 + m_2 l_1 r_2 & I_2 + m_2 r_2^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

We need to also compute the total potential energy stored in the manipulator:

$$U = m_1 g r_1 s_1 + m_2 g (l_1 s_1 + r_2 s_{12})$$

Then we form the Lagrangian:

$$L = T - U$$

and calculate:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \tau$$

We find the dynamic equations of motion as:

$$\begin{bmatrix} I_1 + I_2 + m_1 r_1^2 + m_2(l_1^2 + r_2^2) + 2 m_2 l_1 r_2 c_2 & I_2 + m_2 r_2^2 + m_2 l_1 r_2 \\ I_2 + m_2 r_2^2 + m_2 l_1 r_2 & I_2 + m_2 r_2^2 \end{bmatrix} \ddot{\theta}$$

$$+ \begin{bmatrix} -m_2 l_1 r_2 s_2 \dot{\theta}_2 & -m_2 l_1 r_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ m_2 l_1 r_2 s_2 \dot{\theta}_1 & 0 \end{bmatrix} \dot{\theta} + \begin{bmatrix} m_1 g r_1 c_1 + m_2 g (l_1 c_1 + r_2 c_{12}) \\ m_2 g r_2 c_{12} \end{bmatrix} = \tau$$

To compute the torques necessary to cancel gravity, we can use the principle of virtual work which is essentially the static analysis of the manipulator. Assuming that $\dot{\theta} = \ddot{\theta} = 0$ then we have:

$$\begin{bmatrix} m_1 g r_1 c_1 + m_2 g (l_1 c_1 + r_2 c_{12}) \\ m_2 g r_2 c_{12} \end{bmatrix} = \tau_g$$

The rest of the solution is provided by MATLAB code on the course website.