We said in the previous lecture that our goal was to prove this theorem:

**Theorem:** Consider an $n$ dof serial link robot with no friction at joints and mass-less links with displacements and forces given by:

**Joints**

Displacements: \( \delta q = \begin{pmatrix} \delta q_1 \\ \vdots \\ \delta q_n \end{pmatrix} \)

**End-Effector**

Displacements: \( \delta p = \begin{pmatrix} \delta x_e \\ \delta y_e \\ \delta z_e \\ \delta \phi_x e \\ \delta \phi_y e \\ \delta \phi_z e \end{pmatrix} \)

Forces: \( \mathcal{F} = \begin{pmatrix} f_x \\ f_y \\ f_z \\ N_x \\ N_y \\ N_z \end{pmatrix} \)

where \( J_t \) is the 6x$n$ Jacobian matrix relating infinitesimal joint and end-effector displacements

\[ \delta p = J_t \delta q \]

Then the joint torques \( \tau \) necessary to generate an arbitrary force and moment at the end-effector are given by

\[ \tau = J_t^T \mathcal{F} \]
We also said that the energy method (for now based on the principle of virtual work) is more adequate to study robot manipulators, because of the multiple rigid bodies and multiple constraints:

**PRINCIPLE of VIRTUAL WORK:**

Consider \( m \) forces \( f_1, \ldots, f_m \) acting at \( m \) points \( x_1, \ldots, x_m \) of an \( n \) dof mechanical system.

The virtual work produced by these forces along some reconfiguration:

\[
\delta W_{\text{Work}} = \sum_{j=1}^{m} f_j^T \cdot \delta x_j
\]

where there are a set of virtual displacements of the application points, compatible with the constraints of the mechanism.

The principle of virtual work says that the system will be in "equilibrium" if the virtual work (total) is zero for any possible displacement.

For a kinematic chain:

Let \( q_1, \ldots, q_n \) be a **COMPLETE and INDEPENDENT** set of generalized coordinates.
Then:  

1. Kinematics of the application points:
   \[ x_1 = x_1(q_1, \ldots, q_n) \]
   \[ \vdots \]
   \[ x_m = x_m(q_1, \ldots, q_m) \]

2. Virtual displacements of the application points:
   \[ \delta x_j = \sum_{i=1}^{n} \frac{\partial x_j}{\partial q_i} \delta q_i \]

3. Virtual work:
   \[ \delta W = \sum_{j=1}^{m} f_j \delta x_j = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} f_{j} \frac{\partial x_j}{\partial q_i} \right) \delta q_i \]
   \[ = \sum_{i=1}^{n} \sum_{j=1}^{m} f_{j} \frac{\partial x_j}{\partial q_i} \delta q_i \]
   \[ = Q_1 \delta q_1 + Q_2 \delta q_2 + \cdots + Q_n \delta q_n = 0 \]

   This expression has to be zero \( \forall \delta x_j \)

   \[ Q_i = \sum_{j=1}^{m} f_{j} \frac{\partial x_j}{\partial q_i} = 0 \quad \forall i \]

   Because \((q_1, \ldots, q_n)\) are independent and complete.

Theorem: If we go back now to the theorem

\[ \delta W = z_1 \delta q_1 + \cdots + z_n \delta q_n - F_x \delta x_e - \cdots - N_z \delta \theta_z = \]
\[ = (z^T \delta q - H^T \delta p) = \{ \delta \} \]
\[ = (z^T - H^T J) \delta q = 0 \quad \forall \delta q. \]

Then \( z^T = H^T J \) \[ \rightarrow \]
\[ z = J^T \frac{1}{||} \]

\[ \text{transpose everything} \]
To further explore the duality between differential kinematics and statics, we will look at what happens with singular configurations.

Let's look at the simple manipulator from the previous lecture:

![Manipulator diagram](image)

In configuration $(\theta_1=0^\circ, \theta_2=0^\circ)$

The Jacobian looks like:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{bmatrix}
0 & 0 \\
\ell_1 + \ell_2 & \ell_2
\end{bmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{pmatrix}
\]

→ From a differential kinematics perspective, the domain of possible velocities loses dimension, since both column vectors are parallel to each other.

![Parallel vectors](image)

→ The manipulator is in a singularity since det (J) = 0.

→ Remember also, Nullspace (J) is the set of joint velocities that do not produce a velocity on the end effector.

\[
\begin{bmatrix}
0 & 0 \\
\ell_1 + \ell_2 & \ell_2
\end{bmatrix}
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{pmatrix} = \begin{pmatrix} 0 \\
0
\end{pmatrix}
\]

\[(\ell_1 + \ell_2) \dot{\theta}_1 + \ell_2 \dot{\theta}_2 = 0 \rightarrow \dot{\theta}_2 = -\frac{\ell_1 + \ell_2}{\ell_2} \dot{\theta}_1\]

Nullspace (J) = \langle (\ell_1, -\frac{(\ell_1 + \ell_2)}{\ell_2}) \rangle

→ What happens if we look at that configuration from the perspective of forces?
\[
\begin{bmatrix}
0 & l_1 + l_2 \\
0 & l_2
\end{bmatrix}
\begin{bmatrix}
f_x \\
f_y
\end{bmatrix}
\]

- For any given external force, \( \mathbf{z} \) are uniquely determined.
- Conversely, for a given set of joint torques, it is not always the case that they can be balanced with an external force.

In our case \( z_1 = (l_1 + l_2) f_y \) \( z_2 = l_2 f_y \), we need both torques to balance \((f_x, f_y)\).

But we actually need zero torque to balance \((f_x, 0)\).

![Diagram](image)
The structure bears the force.

Now Nullspace \((J^T)^R\) is the set of end-point forces that do not require joint torques to bear them.

(the structure of the manipulator does it.)

\[
\begin{bmatrix}
0 & l_1 + l_2 \\
0 & l_2
\end{bmatrix}
\begin{bmatrix}
f_x \\
f_y
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\rightarrow f_y = 0, f_x \text{ any value}
\]

So, in summary:

- \( \text{null } (\mathbf{J}) = \text{Set of joint velocities that do not produce end-effector motion} \)
- \( \text{null } (\mathbf{J}^T) = \text{Set of external forces that do not require joint torques to balance} \)
Note precisely:

- In the direction in which joint velocities do not cause any end-effector velocity (Null J1)

  \[ \downarrow \]

  Joint torques cannot be balanced by end-effector forces

- In the direction in which the end-effector cannot be moved

  \[ \downarrow \]

  External forces require no torque to be balanced (Null J1?)

Note that this happens when the Jacobian is degenerate or the arm is in a singular configuration (\( \text{det} = 0 \))

\[ \downarrow \]

In this sense, singularities can be:

- Bad = Cannot control magnitude of force along a certain dimension

- Good = E.g., can carry large loads without relying on strong motors.

The same constraint that does not let the robot move, provides structural support to balance forces.
**EXAMPLE: Deburring robot.**

We want to find out if there are configurations where it is capable of exerting large forces in the vertical direction.

Mechanism 3 dof ($\theta_1, \theta_2, \theta_3$)

\[
\text{dist} (O_0, O_4) = l_0 \\
\text{dist} (O_4, O_5) = l_1 \\
\text{dist} (O_2, O_3) = l_2 \\
\text{dist} (O_3, P) = l_3
\]

Note that:
- Axes $z_0$ and $z_4$ are $\perp$.
- Axes $z_1$ and $z_2$ are $\perp$.
(Which tends to be a good property for resisting loads.)

\[x_P = l_2(\cos \theta_1 \cos \theta_2 \cos \theta_3 - \sin \theta_1 \sin \theta_2) + l_1 \cos(\theta_0) \cos \theta_2 - l_3(\cos \theta_0 \cos \theta_2 \sin \theta_3 + \sin \theta_1 \cos \theta_3)\]

\[y_P = l_2(\sin \theta_1 \cos \theta_2 \cos \theta_3 + \cos \theta_1 \sin \theta_2) + l_1 \sin(\theta_0) \cos \theta_2 - l_3(\sin \theta_1 \cos \theta_2 \sin \theta_3 - \cos \theta_1 \cos \theta_3)\]

\[z_P = l_2(\sin \theta_2 \cos \theta_3) + l_1 \sin \theta_2 + l_0 - l_3 \sin \theta_1 \sin \theta_2 \sin \theta_3\]
The constitutive relationship between forces and motor torques is:

$$
\begin{pmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{pmatrix} = J^T
\begin{pmatrix}
  F_X \\
  F_Y \\
  F_Z
\end{pmatrix}
$$

In general, any solutions to $J^T \begin{pmatrix} F_X \\ F_Y \\ F_Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ will give us forces that the robot can bear.

We want, in particular, forces along $z$, so structurally:

$$
J^T \begin{pmatrix} 0 \\ 0 \\ F_Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

which means that

$$
\begin{pmatrix}
  \frac{\partial z_1}{\partial \theta_1} F_Z \\
  \frac{\partial z_2}{\partial \theta_2} F_Z \\
  \frac{\partial z_3}{\partial \theta_3} F_Z
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

$\forall F_Z \rightarrow \frac{\partial z_1}{\partial \theta_1} = \frac{\partial z_2}{\partial \theta_2} = \frac{\partial z_3}{\partial \theta_3} = 0$

more explicitly:

$$
\frac{\partial z_1}{\partial \theta_1} = 0 \quad \checkmark
$$

$$
\frac{\partial z_2}{\partial \theta_2} = l_2 \cos \theta_2 \cos \theta_3 + l_1 \cos \theta_2 - l_3 \cos \theta_2 \sin \theta_3 =
$$

$$
= \cos \theta_2 \left( l_2 \cos \theta_3 + l_1 - l_3 \sin \theta_3 \right) = \begin{cases} \text{assume } l_1 = l_2 = l_3 = a \end{cases} =
$$

$$
= \cos \theta_2 \left( 1 + \cos \theta_3 - \sin \theta_3 \right) = 0
$$

$$
\rightarrow \cos \theta_2 = 0 \quad \Rightarrow \quad \theta_2 = \pm 90^\circ
$$

$$
\rightarrow (1 + \cos \theta_3 - \sin \theta_3) \Rightarrow \theta_3 = 90^\circ \text{ or } 180^\circ
$$
\[
\frac{d^2 \theta_3}{d\theta_3^2} = -l_2 \sin \theta_2 \sin \theta_3 - l_3 \sin \theta_2 \cos \theta_3 = \\
\sin \theta_2 \left( l_2 \sin \theta_3 + l_3 \cos \theta_3 \right) = \begin{cases} 0 \text{ [assume } l_1 = l_2 = l_3 = 1] \end{cases} \\
\sin \theta_2 \left( \sin \theta_3 + \cos \theta_3 \right) = 0 \\
\rightarrow \sin \theta_2 \Rightarrow \theta_2 = 0^\circ \text{ or } 180^\circ \\
\rightarrow \left( \sin \theta_3 + \cos \theta_3 \right) = 0 \Rightarrow \theta_3 = 135^\circ \text{ or } 315^\circ
\]

The two options are:

(A) \( \theta_2 = \pm 90^\circ \) and \( \theta_3 = 135^\circ \text{ or } 315^\circ \)

(B) \( \theta_2 = 0^\circ, 180^\circ \) and \( \theta_3 = 90^\circ \text{ or } 180^\circ \)

Note that, as expected, in both cases the robot cannot move in the z-direction.