In the previous lecture we saw the Newton's-Euler method to derive the dynamic equations of motion of a robot manipulator.

The Lagrangian is an alternative formulation/principle to describe the evolution of a dynamic system, which instead of balancing force and change in momentum, balances work and change in energy.

Recall: Given a dynamic system
- $n$ degrees of freedom
- $n$ independent and complete generalized coordinates
- $T$ is kinetic energy system. ($q_1, \ldots, q_n$)
- $V$ is potential energy system.

then:
\[
\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{T}}{\partial q_i} = -\frac{\partial V}{\partial q_i} + \mathcal{F}_i
\]

$\mathcal{F}_i$: 1, ..., $n$
Note that both formulations, when applicable, will lead to exactly the same equation of motion.

We will review it in the context of the same example in the previous lecture, the RP 2-link manipulator:

\[ \begin{align*}
&\cdot z_2, \text{ distance from vertical axis to center of mass link 2.} \\
&\cdot \theta_1, \text{ angular orientation link 1.}
\end{align*} \]

\[ \text{STEP 1} \]

Determine a complete and independent set of Generalized Coordinates:

\[ \text{NOTE: Independent: You can fix any n-1 coordinates at any configuration, and change the others freely. Complete: You can specify uniquely all configurations.} \]

Usually these will be the actuator variables. In our case (\(\theta_1, z_2\)).

\[ \text{STEP 2} \]

Compute the Kinetic energy of the system as a function of the generalized coordinates:

In our case:

\[ \text{link}_1: \quad T_1 = \frac{1}{2} I_1 \dot{\theta}_1^2 \text{ (Pure rotation)} \]

\[ \text{link}_2: \quad \text{remember} \]

\[ \begin{align*}
\begin{pmatrix} x_{c_2} \\ y_{c_2} \end{pmatrix} &= J(\theta) \begin{pmatrix} \dot{\theta} \\ \dot{\theta} \end{pmatrix} \\
&= \begin{bmatrix} -2 \sin \theta \cos \theta \\ 2 \cos \theta \sin \theta \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\theta} \end{pmatrix}
\end{align*} \]

Then:

\[ T_2 = \frac{1}{2} m_2 (x_{c_2}^2 + y_{c_2}^2) + \frac{1}{2} I_2 \dot{\theta}_1^2 \]

\( \text{LINEAR ANGULAR} \)
or we can directly write:

\[ T_2 = \frac{1}{2} m_2 \dot{z}^2 + \frac{1}{2} \frac{(I_2 + m_2^2 I_2)}{x^2} \dot{\theta}^2 \]

Inertia about origin

(Using parallel axis theorem)

Then:

\[ T = T_1 + T_2 = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{m_2 \dot{z}^2}{x^2} + \frac{1}{2} I_2 \dot{\Theta}^2 + \frac{1}{2} m_2 \dot{\theta}^2 \]

\[ = \frac{1}{2} \left( I_1 + I_2 + m_2^2 \right) \dot{\theta}^2 + \frac{1}{2} m_2 \dot{\theta}^2 \]

Effective angular inertia

Linear inertia

\[ \text{NOTES: } - T \text{ is in general a function of generalized coordinates and generalized velocities.} \]

\[ \text{The "effective inertia" depends on the configuration of the system, so it depends on } (q_1, \ldots, q_n) \]

\[ \text{different inertial properties.} \]

\[ - T \text{ can always be written as a quadratic form of the generalized velocities:} \]

\[ T = \frac{1}{2} \left( \dot{\theta} \right)^T \left[ \begin{array}{cc} I_1 + I_2 + m_2 \dot{z}^2 & 0 \\ 0 & m_2 \end{array} \right] \left( \dot{\theta} \right) \]

\[ \text{or } \quad T = \frac{1}{2} \dot{\theta}^T H \dot{\theta} \quad \text{(generalized version of } \frac{1}{2} m \dot{r}^2) \]
Where $H(q)$ is the INERTIA MATRIX

Inertial (effective) that generalized coordinates see.

**STEP 3** Compute the Potential Energy of the system as a function of the generalized coordinates.

In our case, there are no conservative forces (springs, gravity,...) so $U(q) = 0$

**NOTES:** $U$ is in general a function of the generalized coordinates $(q)$ and not of the generalized velocities $(\dot{q})$

- A typical example is to use springs to partially fix the gravity:

![Diagram showing two arms with and without springs.](attachment:diagram.png)

**STEP 4** Find virtual work produced by non-conservative forces and derive generalized forces:

For a system with $m$ forces $f_i$ applied at points $x_i$:

$$\delta W = \sum_{i=1}^{m} f_i \cdot \delta x_i$$

any motion of the application point consistent with the constraints of the system

then we rewrite it as:

$$\delta W = \sum_{i=1}^{n} Q_i \cdot \delta q_i$$

identify these gen. forces.
In our case (and in many cases) we only have to worry about actuator forces \((\tau_1, f_2)\) which happen to be well aligned with the selection of generalized coordinates:

\[
\Theta (\text{fixed } f) : \quad \delta W_\Theta = 2 \tau_1 \delta \Theta
\]

\[
\delta \theta (\text{free } f) \quad \delta W_\theta = \left(f_2 \cos \theta_1, f_2 \sin \theta_1 \right) \cdot \left(\delta \theta \cos \theta_1, \delta \theta \sin \theta_1 \right) = f_2 \delta \theta
\]

In total:

\[
\delta W = 2 \tau_1 \delta \Theta + f_2 \delta \theta
\]

\[
Q_1 = \tau_1, \quad Q_2 = f_2
\]

In general it might be a bit more complex if forces are not aligned with generalized coordinates, and their expression will change:

For example, consider the RR manipulator:

\[
\text{case 1:} \quad \delta \theta_1 \quad \delta W_1 = \tau_1 \delta \theta_1
\]

\[
\delta \theta_2 \quad \delta W_2 = \tau_2 \delta \theta_2
\]

\[
Q_1 = \tau_1, \quad Q_2 = f_2
\]
now we redefine the generalized coordinates so both angles are 
absolute, not relative:

\[ \text{case 2: } \begin{align*} 
& \delta \Theta_1 \quad \text{(fixed)} \\
& \delta \Theta_2 \quad \text{(fixed)} 
\end{align*} \]

so that \( \delta \Theta_2 \) does not change, actuator 2 
has to move backwards the same as \( \delta \Theta_1 \):

\[ \delta W_1 = \delta \Theta_1 \cdot z_1 = \delta \Theta_1 \cdot z_2 
\]

\[ \delta W_2 = \delta \Theta_2 \cdot z_2 \]

In total:

\[ \delta W = \delta \Theta_1 \cdot z_1 + \delta \Theta_2 \cdot z_2 + \delta \Theta_1 \cdot z_2 = 
(\delta \Theta_1 \cdot z_1 - \delta \Theta_1 \cdot z_2) + \delta \Theta_1 \cdot z_2 = 
= (z_1 - z_2) \delta \Theta_1 + z_2 \delta \Theta_2 \]

\[ Q_1 = (z_1 - z_2) \delta \Theta_1 \quad Q_2 = z_2 \delta \Theta_2 \]

and further consider a third situation where we do not change 
the generalized coordinates but the geometry of actuation:

\[ \text{case 3: } \begin{align*} 
& \delta \Theta_1 \quad \text{and } \delta \Theta_2 \\
& \text{both actuators are now at the base and one transmits actuation} \\
& \text{with a belt.} 
\end{align*} \]
\[ \delta W_1 = \delta \theta_1 \cdot 2_z + \delta \theta_4 \cdot 2_z \]
\[ \delta W_2 = \delta \theta_2 \cdot 2_z \]
\[ \delta W = (2_1 + 2_2) \delta \theta_1 + 2_2 \delta \theta_2 \]
\[ Q_1 = (2_1 + 2_2) \]
\[ Q_2 = 2_2 \]

**STEP 5**

Finally, we use Euler–Lagrange's equation:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{\partial V}{\partial q_i} + \sum_{j=1}^{n} \frac{\partial q_j}{\partial q_i} \dot{q}_j \quad i = 1, \ldots, n
\]

**MOTIONS**

\[ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{\partial V}{\partial q_i} \]

**FORCES**

Effectively is a relationship between motions and forces, but in the space of gen. coordinates.

In our case:

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} = \frac{d}{dt} (2 m_2 \dot{z}) = m_2 \ddot{z}
\]

\[
\frac{\partial T}{\partial q_1} = 0 \quad Q_1 = 2_1
\]

\[
\frac{\partial T}{\partial q_2} = m_2 \dot{z} \dot{\theta}^2 \quad Q_2 = f_2
\]
so far \( i = 1 \) \( (q_1 = 0) \) \[ \dot{2}m_2 \dot{z} \ddot{\theta} + (I_1 + I_2 + m_2 \dot{z}^2) \ddot{\theta} = f_1 \]

\( i = 1 \) \( (q_2 = \ddot{z}) \) \[ m_2 \ddot{z} - m_2 \dot{z} \dot{\theta}^2 = f_2 \]

Identical to the equations we got through the Newton-Euler approach.

**NOTES:** → We do not need to explicitly make use of the constraint forces or moments. These are already implicit in the selection of generalized coordinates.

→ This is key. For example, a pin-joint in 3D requires 5 constraints for forces/torques!

→ Unfortunately, this means that Lagrange is only applicable to **HOLONOMIC** systems (\#DOFs = \#G.C.)

For example, a wheeled vehicle with no slip on the ground has to be solved with Newton-Euler.

(The are more constraints that limit the DOFs than those induced by the generalized coordinates.)