Problem 1 (35 points)

Inverse kinematics of an RPR Manipulator

The planar robot in Figure 1 is comprised of two revolute joints ($\theta_1, \theta_3$) and one prismatic joint ($d_2$). The first link has a $90^\circ$ bend, so that the prismatic axis is always at a constant distance $l_1$ from O. The end effector is a point at a constant distance $l_3$ from the second revolute joint, with coordinates $(x_e, y_e)$ and orientation $\phi_e$ with respect the horizontal.

![Figure 1: RPR Manipulator.](image)

a. Obtain the kinematic equations that describe the end-effector position $(x_e, y_e)$ and orientation $\phi_e$ as a function of joint displacements $(\theta_1, d_2, \theta_3)$.

b. Assume now that the joints are limited to these ranges:

$$\theta_1 \in [45^\circ, 135^\circ] \quad d_2 \in [l_1, 2l_1] \quad \theta_3 \in [-90^\circ, 90^\circ]$$

Sketch the workspace of the end-effector in the $XY$ plane (do not consider $\phi_e$).

c. Find the expression for the inverse kinematics of the manipulator (relating $(x_e, y_e, \phi_e)$ to $(\theta_1, d_2, \theta_3)$).
**Part a: (10 points)** This is the problem of the forward kinematics of the manipulator and the expressions for the end effector are given by:

\[
x_e = l_1 \cos \theta_1 + d_2 \cos(\theta_1 - \pi/2) + l_3 \cos(\theta_1 + \theta_3 - \pi/2) \\
y_e = l_1 \sin \theta_1 + d_2 \sin(\theta_1 - \pi/2) + l_3 \sin(\theta_1 + \theta_3 - \pi/2) \\
\phi_e = \theta_1 + \theta_3 - \pi/2
\]

or

\[
x_e = l_1 \cos \theta_1 + d_2 \sin(\theta_1) + l_3 \sin(\theta_1 + \theta_3) \\
y_e = l_1 \sin \theta_1 + d_2 \cos(\theta_1) + l_3 \cos(\theta_1 + \theta_3) \\
\phi_e = \theta_1 + \theta_3 - \pi/2
\]

**Part b: (10 points)**

![Figure 2: RPR Manipulator workspace.](image)

**Part c: (15 points)** Labels:

- 90 degree elbow - A
- 3rd joint - B

We will use a geometric approach to find the inverse kinematics of the manipulator. Given the end effector position and orientation point B is fully defined and the coordinates are given by:

\[
\begin{bmatrix}
x_b \\
y_b
\end{bmatrix} = \begin{bmatrix}
l_3 - \cos(\phi_e) \\
l_e - \sin(\phi_e)
\end{bmatrix}
\]

Next carefully inspect the triangle OAB, using Pythagoras theorem:

\[
OB^2 = OA^2 + AB^2 \\
x_b^2 + y_b^2 = d_2^2 + l_1^2 \\
d_2^2 = x_b^2 + y_b^2 - l_1^2 \\
d_2 = \sqrt{(l_3 - \cos \phi_e)^2 + (l_e - \sin \phi_e)^2 - l_1^2}
\]
Next to find the angle for the first joint we can write:

\[ \theta_1 = \tan^{-1}\left(\frac{y_b}{x_b}\right) + \tan^{-1}\left(\frac{\sqrt{x_b^2 + y_b^2 - l_1^2}}{l_1}\right) \]

and finally to find \( \theta_3 \) we can use

\[ \theta_3 = \phi_e + \pi/2 - \theta_1 \]

For this problem algebraic approaches are also acceptable using the laws of cosine, i.e. we can write:

\[ \theta_1 = \tan^{-1}\left(\frac{y_b}{x_b}\right) + \cos^{-1}\left(\frac{l_1}{\sqrt{x_b^2 + y_b^2}}\right) \]

**Problem 2 (25 points)**

**Standard reference frames of a robotic manipulator**

As a matter of convention, it is helpful to give particular names to frequent reference frames associated with a robot manipulator and its workspace. Figure 3 shows a typical situation in which a robot is carrying out a task on a table, with six common frames:

- **World frame** \( O \)  Inertial reference frame.
- **Base frame** \( B \)  Located at the base of the robot manipulator.
- **Station frame** \( S \)  Located in the vicinity of the relevant working area. It simplifies the description of the task. It is usually specified in the world frame \( OT \).
- **Wrist frame** \( W \)  Affixed to the last link of the manipulator. It is usually specified in the base frame, encoding the direct kinematics of the robot arm \( BT \).
- **Tool frame** \( T \)  Also known as end-effector frame, is located at a useful point and orientation in the tool, e.g., its tip. It is always specified in the frame \( WT \).
- **Goal frame** \( G_i \)  Location where the tool has to go to execute task \( i \). It is usually specified in the station frame \( ST \).

![Figure 3: (left) Standard reference frames; (right) Calibration socket.](image)

a. **Calibration.** A normal required step for proper use of a tool is to calibrate its location with respect to the wrist of the robot \( WT \). With that intention, we manually guide the robot in order to insert
the tool into the calibration socket. Once in this calibration configuration (in which frames \( G_1 \) and \( T \) are coincident), we read the position of the robot with respect to its base using direct kinematics \( B_W T \). Assuming that the locations of the station \( O_S T \) and calibration goal \( S_{G_1} T \) are known, and that the location of the base \( B_T \) is given by some localization system, find the expression of the calibration tool frame \( W_T T \) as a function of other known transformations.

b. **Station coordinates.** It is often also useful to figure out the location of the tool with respect to the station coordinates \( S_T \). If the robot is in a particular configuration given by the location of its base \( O_B T \) and its manipulator \( B_W T \), and if we have already calibrated the tool \( W_T T \), and if the location of the station is known \( O_S T \), figure out the transformation \( S_T T \) that gives the location of the tool in station coordinates.

c. In practice, some of the expressions above might require to invert a matrix. Should we expect problems when inverting them?

**Part a: (10 points)** We can write:

\[
\begin{align*}
O_{G_1} T &= O_S T \cdot S_{G_1} T \\
O_W T &= O_B T \cdot B_W T \\
W_T T &= W_{G_1} T
\end{align*}
\]

Consequently:

\[
W_T T = W_{G_1} T = O_W T^{-1} O_{G_1} T
\]

**Part b: (10 points)** We can write:

\[
S_T T = (O_S T)^{-1} O_B T \cdot B_W T \cdot W_T T
\]

**Part c: (5 points)** No there is no problem, the homogeneous transformations are always invertible.

**Problem 3 (40 points)**

**Manipulator with long reach**

The manipulator in Figure 4 has 2 degrees of freedom \((\theta_1, \theta_2)\). As it stands the manipulator’s end effector \( A \) cannot reach the point \( B \) because \( l_3 > l_1 + l_2 \). To extend its workspace, we propose to mount it on a linear guide, allowing joint one to move linearly along axis X, where the distance from the origin is denoted by \( d \), as in Figure 5. With this in mind answer the following questions:

a. Derive the **forward kinematics** and the **Jacobian** of the manipulator in Figure 4 relating \((\theta_1, \theta_2)\) to the Cartesian position of \( A = (x_A, y_A) \).

b. Construct the **Jacobian** of the new manipulator in Figure 5 relating now \((\dot{d}, \dot{\theta}_1, \dot{\theta}_2)\) to \((\dot{x}_A, \dot{y}_A, \dot{\phi})\), where \( \phi \) is the orientation of the last link with respect the horizontal. Describe the effect on the Jacobian from adding the guide as an additional degree of freedom.

c. Compute the **singular configurations** of both manipulators. Sketch the manipulator in those configurations.

d. For this problem assume that \( l_1 = l_2 = 1 \) and \( l_3 = 2 \), and that the robot in Figure 5 is in configuration \( \theta_1 = -\pi/4, \theta_2 = \pi/2 \) with the end effector at \( B \). If we only care about the position of the end effector and not its orientation, is the robot **redundant**? If it is, describe the set of joint velocities \((\dot{d}, \dot{\theta}_1, \dot{\theta}_2)\) that produce zero velocity in the end effector \((\dot{x}_A, \dot{y}_A)\).
Part a: (10 points) Forward kinematics are given by:
\[
\begin{bmatrix}
    x_A \\
    y_A
\end{bmatrix} = \begin{bmatrix}
    l_1 c_1 + l_2 c_{12} \\
    l_1 s_1 + l_2 s_{12}
\end{bmatrix}
\]
and the Jacobian is found to be:
\[
J = \begin{bmatrix}
    -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\
    l_1 c_1 + l_2 c_{12} & l_2 c_{12}
\end{bmatrix}
\]

Part b: (10 points) The forward kinematics with the new degree of freedom is:
\[
\begin{bmatrix}
    x_A \\
    y_A \\
    \phi
\end{bmatrix} = \begin{bmatrix}
    d + l_1 c_1 + l_2 c_{12} \\
    l_1 s_1 + l_2 s_{12} \\
    \theta_1 + \theta_2
\end{bmatrix}
\]
The new Jacobian is:
\[
J = \begin{bmatrix}
    1 & -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\
    0 & l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\
    0 & 1 & 1
\end{bmatrix}
\]
The new Jacobian has an additional column due to the new and independent degree of freedom and a row to represent the variation in the angle of the end effector.
Part c: (10 points) We already know that the singular configurations of the RR manipulator are given by $\theta_2 = 0, \pi$. The singular configurations for the new manipulator are given at $\theta_1 = \pi/2, -\pi/2$ which are found from deriving the determinant which is given by $\det(J) = l_1 c_1 = 0$. The singular configurations are shown below and are fundamentally different types.

Part d: (10 points) In this part our forward kinematics is simply the first two rows of the forward kinematics in the previous section:

$$
\begin{bmatrix}
  x_A \\
  y_A
\end{bmatrix} =
\begin{bmatrix}
  d + l_1 c_1 + l_2 c_{12} \\
  l_1 s_1 + l_2 s_{12}
\end{bmatrix}
$$

We need to compute the Jacobian and find its null space to evaluate the redundancy of the manipulator:

$$
J = \begin{bmatrix}
  1 & -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\
  0 & l_1 c_1 + l_2 c_{12} & l_2 c_{12}
\end{bmatrix}
$$

Next we apply the assumptions provided in the problem statement:

$$
J = \begin{bmatrix}
  1 & 0 & -\frac{1}{\sqrt{2}} \\
  0 & \sqrt{2} & \frac{1}{\sqrt{2}}
\end{bmatrix}
$$

Next we find solutions to:

$$
J \begin{bmatrix}
  \dot{d} \\
  \dot{\theta}_1 \\
  \dot{\theta}_2
\end{bmatrix} = 0
$$

Clearly there are 3 unknowns and only 2 equations, we can parametrize the solution using $\dot{\theta}_2$:

$$
N = \begin{bmatrix}
  1 \\
  -\frac{\sqrt{2}}{2} \\
  \frac{1}{2}
\end{bmatrix} \dot{\theta}_2
$$