Massachusetts Institute of Technology
Department of Mechanical Engineering

2.12 Introduction to Robotics
Solutions of the End-of-Term Examination
Fall 2008

Problem 1 (40 points)

A new underactuated carnival ride is shown in Figure 1 ***a*** and ***b*** below. (Do not try this. It is dangerous). A long beam of length ℓ_1, *Link 1*, is free to rotate about *Joint 1*, with a small viscous damping torque proportional to joint velocity $\dot{\theta}_1$, i.e. $-b\dot{\theta}_1$. Note that the joint axis is ***tilted*** angle ϕ as shown in Figure 1-***a***. A large rotary disk, *Link 2*, rotates about *Joint 2* with actuator torque $\boldsymbol{\tau}$ generated by a hydraulic actuator placed at the tip of *Link 1*. Passengers sit on the rotary disk with seat belts fastened. They are paired so that the center of mass of the entire passengers and the rotary disk is approximately at Joint 2. The total mass of the passengers and the disk is m_2 and the moment of inertia about the center of mass is I_2. *Link 1* weighs m_1 and its center of mass is at distance ℓ_{C1} from *Joint 1* with moment of inertia I_1 at the center of mass. Joint angles θ_1 and θ_2 are used as generalized coordinates. Note that angle θ_2 is measured from a horizontal line parallel to axis ***x***, as shown in Figure 1-***b***. Answer the following questions.

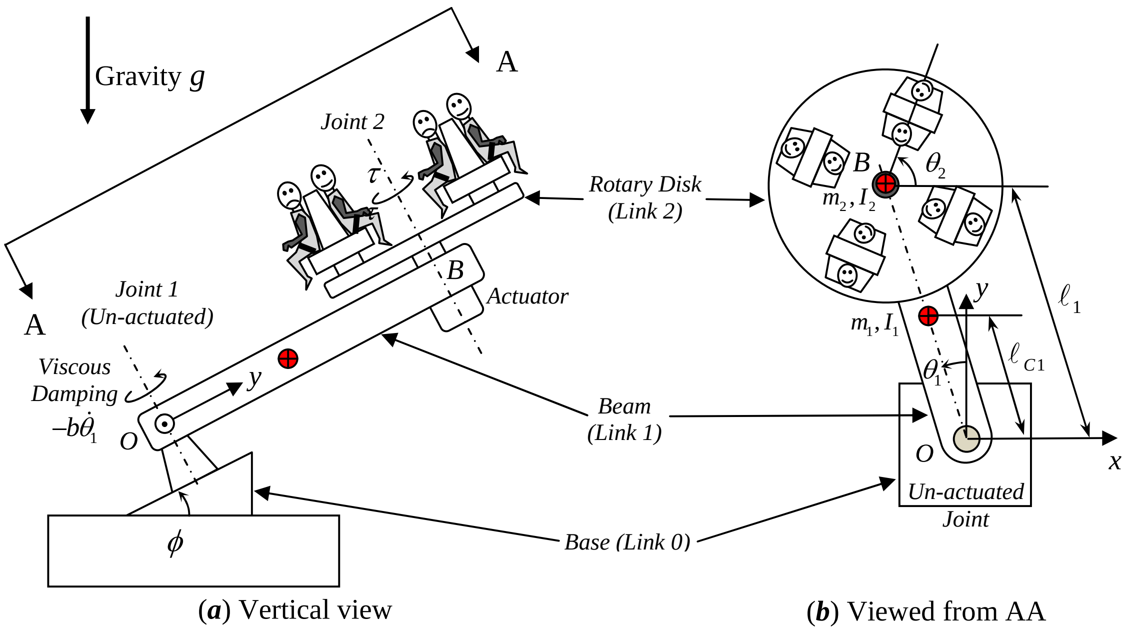


Figure 1 Under-actuated carnival ride

a). Obtain kinetic energy T of the entire system using the generalized coordinates.
b). Obtain generalized forces Q_1 and Q_2 associated with generalized coordinates θ_1 and θ_2, respectively.
c). Obtain potential energy U of the entire system using the generalized coordinates.
d). Obtain Lagrange's equations of motion.
e). Linearize the equations of motion in the vicinity of $\theta_1 = 0$.

f). Obtain the transfer function from actuator torque τ to θ_1, $G(s) = \theta_1(s)/\tau(s)$.

g). Draw a root locus for Proportional control of the linearized system, and discuss stability near $\theta_1 = 0$.

h). Draw a root locus for Proportional and Derivative control of the same linearized system, and discuss stability.

Solution

a). From the figure, the squared velocity of each center of mass is $|\mathbf{v}_{C1}|^2 = \ell_{C1}{}^2\dot{\theta}_1^2, |\mathbf{v}_{C2}|^2 = \ell_1{}^2\dot{\theta}_1^2$. Substituting these into the total kinetic energy equation yields:

$$T = T_1 + T_2 = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_1|\mathbf{v}_{C1}|^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_2|\mathbf{v}_{C2}|^2$$

$$= \frac{1}{2}(I_1 + m_1\ell_{C1}{}^2 + m_2\ell_1{}^2)\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

Note that θ_2 is measured from a horizontal line, which is independent of θ_1. Therefore, $\omega_2 = \dot{\theta}_2$, and $\omega_2 \neq \dot{\theta}_1 + \dot{\theta}_2$

b). Consider the free-body diagram of each link. The virtual work done by the actuator torque and the viscous damping on the two links is given by

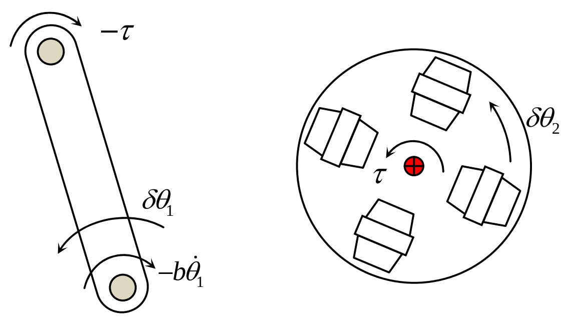
$$\delta Work = (-\tau - b\dot{\theta}_1)\delta\theta_1 + \tau\delta\theta_2$$

Comparing this to

$$\delta Work = Q_1\delta q_1 + Q_2\delta q_2$$

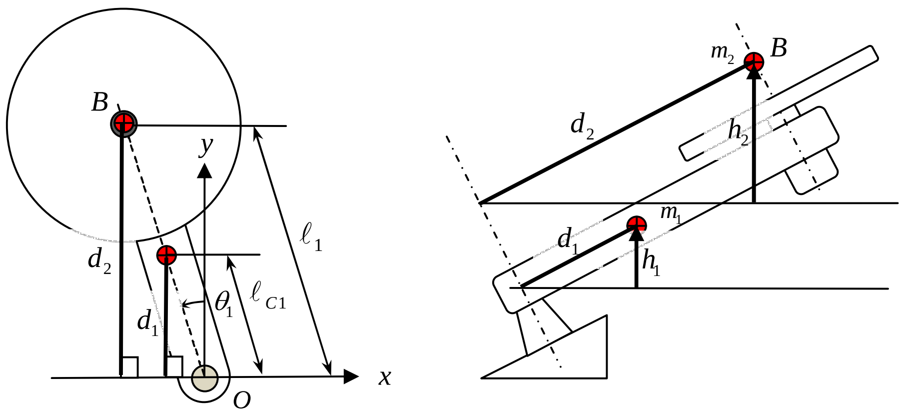
yields

$$Q_1 = -\tau - b\dot{\theta}_1, \quad Q_2 = \tau$$



c). From the diagrams below

$$d_1 = \ell_{C1}\cos\theta_1, d_2 = \ell_1\cos\theta_1, \quad h_1 = d_1\sin\phi, h_2 = d_2\sin\phi$$



The potential energy is then given by

$$U = m_1gh_1 + m_2gh_2 = g\sin\phi(m_1\ell_{C1} + m_2\ell_1)\cos\theta_1$$

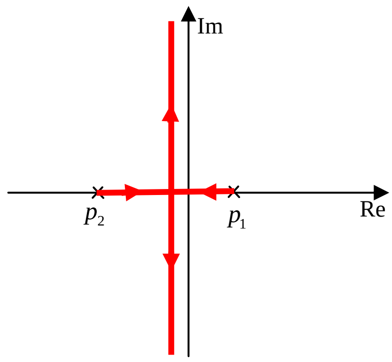
d). From the above results,

$$T = \frac{1}{2}H_{11}\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2, \quad where\ H_{11} \triangleq I_1 + m_1\ell_{C1}{}^2 + m_2\ell_1{}^2$$

$$U = g_0 \cos\theta_1, \quad where\ g_0 \triangleq g \sin\phi(m_1\ell_{C1} + m_2\ell_1)$$
$$Q_1 = -\tau - b\dot{\theta}_1, \quad Q_2 = \tau$$
Substituting these into Lagrange's equations of motion :
$$Q_i - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} = 0, \qquad i = 1,2$$
yields
$$-b\dot{\theta}_1 - \tau - H_{11}\ddot{\theta}_1 + g_0 \sin\theta_1 = 0 \text{, for } q_1 = \theta_1 \text{, and}$$
$$\tau - I_2\ddot{\theta}_2 = 0 \text{, for } q_2 = \theta_2 .$$

e). The only nonlinear term involved Lagrange's equations of motion is
$$g_0 \sin\theta_1 \cong g_0\theta_1$$
Therefore the linearized equations are given by
$$-b\dot{\theta}_1 - \tau - H_{11}\ddot{\theta}_1 + g_0\theta_1 = 0 ,$$
$$\tau - I_2\ddot{\theta}_2 = 0 .$$

f). Taking Laplace transform of the first equation:
$$-bs\theta_1(s) - \tau(s) - H_{11}s^2\theta_1(s) + g_0\theta_1(s) = 0$$
therefore
$$G(s) = \frac{\theta_1(s)}{\tau(s)} = \frac{-1}{H_{11}s^2 + bs - g_0}$$

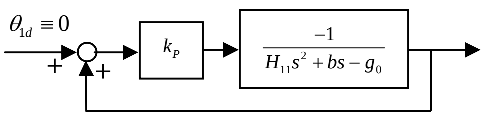


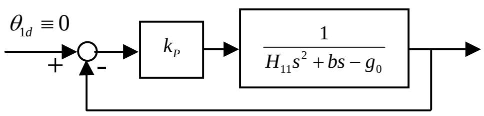
g). Open-loop poles:
Solving the characteristic equation: $H_{11}s^2 + bs - g_0 = 0$, yields
$$p_{1,2} = \frac{1}{2H_{11}}\left(-b \pm \sqrt{b^2 + 4H_{11}g_0}\right).$$
Both poles are real; one positive and one negative.

Check the sign of the real poles:
$b < \sqrt{b^2 + 4H_{11}g_0}$ → $p_1 > 0,\ p_2 < 0$

The open-loop control system is unstable. Proportional control is conditionally stable. See the root locus.It becomes stable for higher gain, but oscillatory.

Note that this feedback is positive feedback, since the sign of the transfer function is negative. Using the equivalent transformation below, this can be converted to the standard negative feedback, as discussed in class.

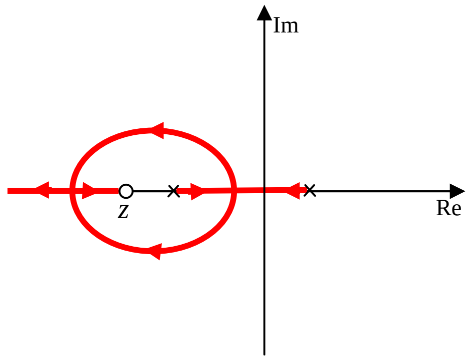




h). PD control adds one zero:

$$k_P(1+k_D s)=0 \rightarrow z=-\frac{1}{k_D}$$

PD control not only stabilizes the system but also increases damping.
See the root locus.



Problem 2 (20 points)

A robot is cleaning a glass window with a rectangular sponge, as shown in Figure 2. The process is assumed to be quasi-static and friction-less, and the robot is controlled based on Mason's hybrid position / force control law. Answer the following questions.

a). Obtain the natural and artificial constraints with respect to the C-frame shown in the figure.

b). Sketch the block diagram of hybrid position/force control system, and obtain the projection matrices, ***P***a and ***P***c, and the reference inputs involved in the control system.

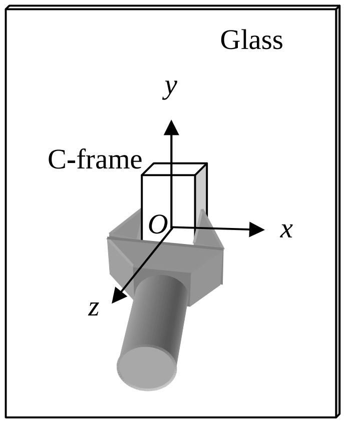
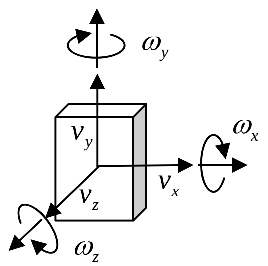


Figure 2 Window cleaning robot

Solutions

a). The sponge has a face-to-face contact with the glass surface. Three degrees of freedom among 6 are constrained by the contact.

	Kinematic	Static
Natural Constraints	$v_z=0$ $\omega_x=0$ $\omega_y=0$	$f_x=0$ $f_y=0$ $\tau_z=0$
Artificial Constraints	$v_x=V_x$ $v_y=V_y$ $\omega_z=0$	$f_z=-F<0$ $\tau_x=0$ $\tau_y=0$



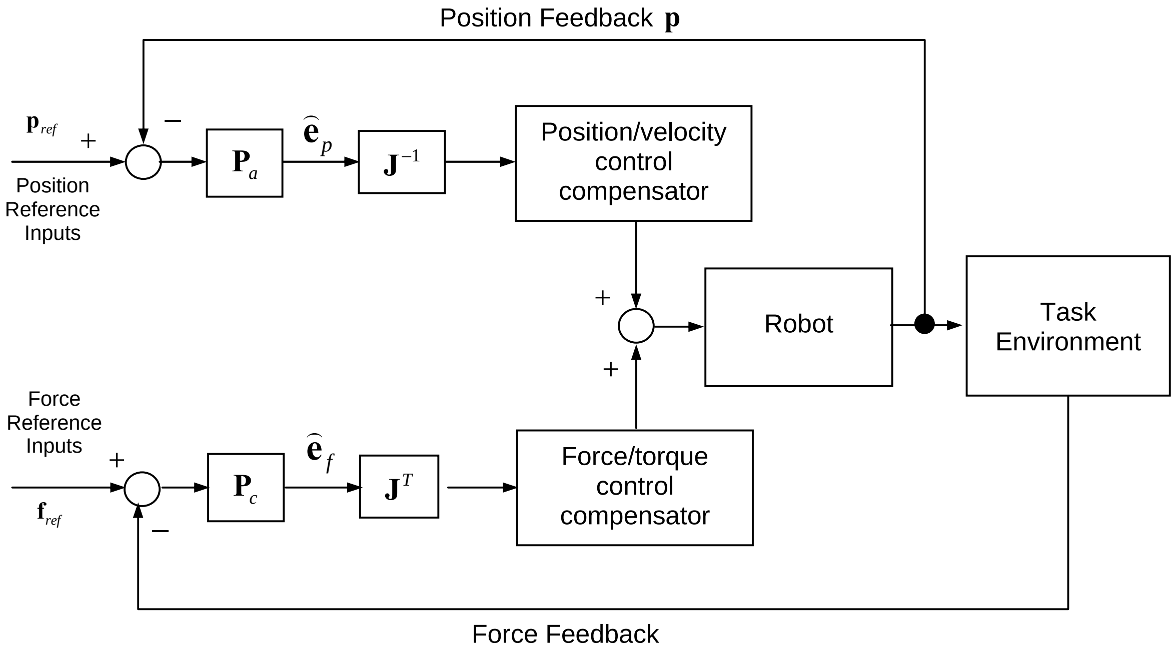
b). The projection matrices are : $\mathbf{P}_a=diag(1,1,0,0,0,1)$, $\mathbf{P}_c=diag(0,0,1,1,1,0)$
The reference inputs;

Position reference

$$\mathbf{p}_{ref} = \begin{pmatrix} V_x \cdot t + x_0 \\ V_y \cdot t + y_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Force reference

$$\mathbf{f}_{ref} = \begin{pmatrix} 0 \\ 0 \\ -F \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

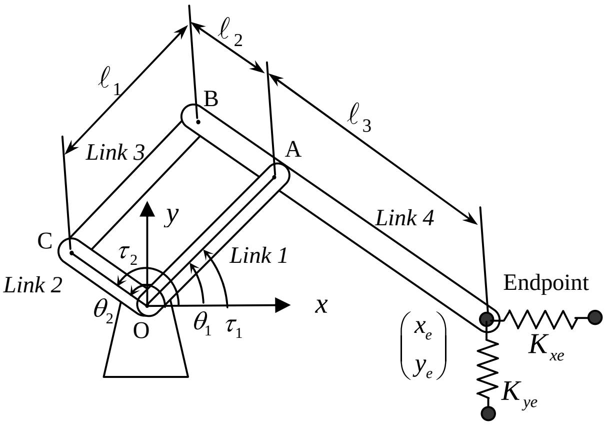


Block diagram of hybrid position/force control system

Problem 3 (40 points)

The photo below is the original Phantom® haptic device with a parallelogram mechanism. As a human operator moves the endpoint, he/she feels a resistive force generated by the actuators driving the linkage. Figure 4 depicts the kinematic structure of the two degree-of-freedom parallelogram mechanism powered by two motors placed at the base. *Motor* 1 rotates *Link* 1, varying joint angle θ_1 , while *Motor* 2 rotates *Link* 2, varying θ_2. The length of *Links* 1 and 3 are the same, ℓ_1. The length of *Link* 2 is ℓ_2, and it is parallel to *Link* 4. The end effecter is placed at distance ℓ_3 from point *A*, as shown in the figure. A Cartesian coordinate system, *O-xy*, is placed at the base. Figure 5 shows mass properties of the individual links denoted by mass m_i, centroidal moment of inertia I_i, and the location of the center of mass ℓ_{Ci}, $i = 1, \cdots, 4$. Answer the following questions.

Figure 3 original Phantom® haptic device



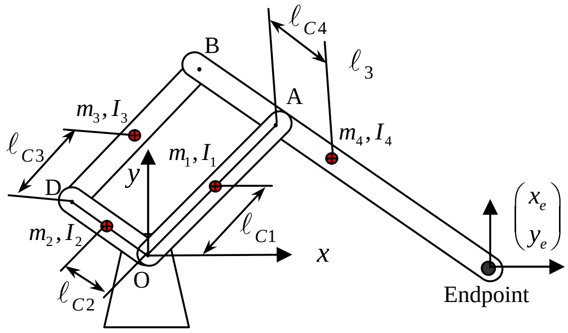


Figure 4 Kinematic structure and endpoint stiffness

Figure 5 Parallelogram mass properties

a). Obtain the Jacobian matrix relating endpoint velocities $\dot{x}_e$ and $\dot{y}_e$ to joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$.
b). As the operator pushes the endpoint, he/she feels some stiffness, as illustrated by two virtual springs, K_{xe} in the *x* direction and K_{ye} in the *y* direction. Obtain the 2 x 2 joint feedback gain matrix $\mathbf{K}_q$ that creates the endpoint stiffness:

$$\mathbf{K}_p = \begin{pmatrix} K_{xe} & 0 \\ 0 & K_{ye} \end{pmatrix} .$$

c). Obtain the kinetic energy stored in the four links when the joints are moving at $\dot{\theta}_1$ and $\dot{\theta}_2$, and find the inertia matrix associated with the joint coordinates θ_1 and θ_2:

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}.$$

d). Discuss the physical sense of components H_{11} and H_{22}. Explain why both H_{11} and H_{22} do not vary depending on the joint angles θ_1 and θ_2.

e). (**Extra Credit**) When the human operator moves the endpoint quickly, he/she feels some inertia of the arm links reflected to the endpoint. Obtain the inertia matrix of the system reflected to the endpoint. (Hint: Use endpoint coordinates x_e and y_e as generalized coordinates, and obtain the inertia matrix with respect to x_e and y_e.)

Solution

a).
$$\begin{aligned} x_e &= \ell_1 \cos\theta_1 + \ell_3 \cos(\theta_2 - \pi) = \ell_1 \cos\theta_1 - \ell_3 \cos\theta_2 \\ y_e &= \ell_1 \sin\theta_1 - \ell_3 \sin\theta_2 \end{aligned}$$

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \quad where\ \mathbf{J} = \begin{pmatrix} -\ell_1 \sin\theta_1 & \ell_3 \sin\theta_2 \\ \ell_1 \cos\theta_1 & -\ell_3 \cos\theta_2 \end{pmatrix}$$

b). $\boldsymbol{\tau} = \mathbf{J}^T\mathbf{F} = \mathbf{J}^T\mathbf{K}_P\Delta\mathbf{p} = \mathbf{J}^T\mathbf{K}_P\mathbf{J}\Delta\boldsymbol{\theta}, \rightarrow \mathbf{K}_q = \mathbf{J}^T\mathbf{K}_P\mathbf{J}$

$$\mathbf{K}_q = \mathbf{J}^T\mathbf{K}_P\mathbf{J} = \begin{pmatrix} -\ell_1 s_1 & \ell_1 c_1 \\ \ell_3 s_2 & -\ell_3 c_2 \end{pmatrix} \begin{pmatrix} K_{xe} & 0 \\ 0 & K_{ye} \end{pmatrix} \begin{pmatrix} -\ell_1 s_1 & \ell_3 s_2 \\ \ell_1 c_1 & -\ell_3 c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \ell_1^2(s_1^2 K_{xe} + c_1^2 K_{ye}) & -\ell_1\ell_3(s_1 s_2 K_{xe} + c_1 c_2 K_{ye}) \\ -\ell_1\ell_3(s_1 s_2 K_{xe} + c_1 c_2 K_{ye}) & \ell_3^2(s_2^2 K_{xe} + c_2^2 K_{ye}) \end{pmatrix}$$

c). Kinetic energy:

$$T = T_1 + T_2 + T_3 + T_4$$

$$= \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m_1|\mathbf{v}_{C1}|^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}m_2|\mathbf{v}_{C2}|^2 + \frac{1}{2}I_3\dot{\theta}_1^2 + \frac{1}{2}m_3|\mathbf{v}_{C3}|^2 + \frac{1}{2}I_4\dot{\theta}_2^2 + \frac{1}{2}m_4|\mathbf{v}_{C4}|^2$$

Computing the centroidal velocities:

$$|\mathbf{v}_{C1}|^2 = \ell_{C1}^2\dot{\theta}_1^2$$

$$|\mathbf{v}_{C2}|^2 = \ell_{C2}^2\dot{\theta}_2^2$$

The centroidal velocities of Link 3 are derived from

$$x_{C3} = \ell_2 c_2 + \ell_{C3} c_1$$

$$y_{C3} = \ell_2 s_2 + \ell_{C3} s_1$$

$$\dot{x}_{C3} = -\ell_2 s_2\dot{\theta}_2 - \ell_{C3} s_1\dot{\theta}_1$$

$$\dot{y}_{C3} = \ell_2 c_2\dot{\theta}_2 + \ell_{C3} c_1\dot{\theta}_1$$

$$|\mathbf{v}_{C3}|^2 = \dot{x}_{C3}^2 + \dot{y}_{C3}^2 = \ell_2^2\dot{\theta}_2^2 + \ell_{C3}^2\dot{\theta}_1^2 + 2\ell_2\ell_{C3}\cos(\theta_2 - \theta_1)\cdot\dot{\theta}_1\dot{\theta}_2$$

Similarly,

$$|\mathbf{v}_{C4}|^2 = \dot{x}_{C4}^2 + \dot{y}_{C4}^2 = \ell_1^2\dot{\theta}_1^2 + \ell_{C4}^2\dot{\theta}_2^2 - 2\ell_1\ell_{C4}\cos(\theta_2 - \theta_1)\cdot\dot{\theta}_1\dot{\theta}_2$$

Substituting these into the kinetic energy and collecting terms yield

$$T = \frac{1}{2}H_{11}\dot{\theta}_1^2 + H_{12}\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}H_{22}\dot{\theta}_2^2$$

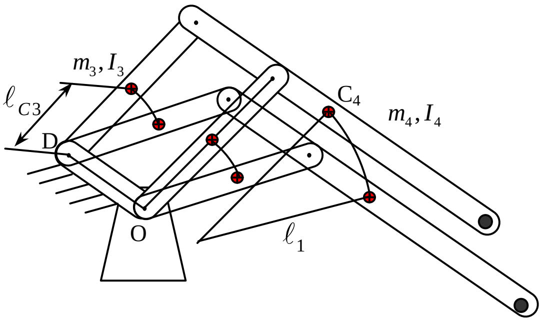
where

$$H_{11} \triangleq I_1 + m_1\ell_{C1}^2 + I_3 + m_3\ell_{C3}^2 + m_4\ell_1^2$$

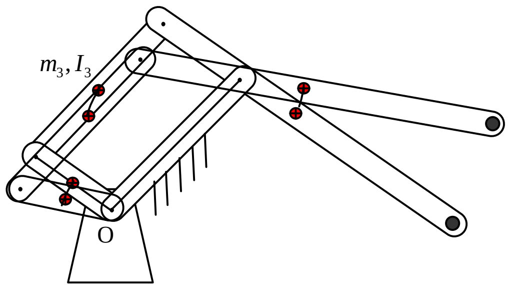
$$H_{22} \triangleq I_2 + m_2\ell_{C2}^2 + I_4 + m_4\ell_{C4}^2 + m_3\ell_2^2$$

$$H_{12} = (m_3\ell_2\ell_{C3} - m_4\ell_1\ell_{C4})\cos(\theta_2 - \theta_1)$$

d). H_{11} is the total inertia seen by the first generalized coordinate θ_1 while the other generalized coordinate θ_2 is fixed. See the figure below. As θ_1 varies, the parallelogram deforms. The motion of Link 1 is a fix-point rotation. Therefore, the moment of inertia associated with the rotation about point O is $I_1 + m_1 \ell_{C1}^{\ 2}$, according to the Parallel Axis Theorem. This does not vary depending on the value of θ_2. Link 3, too, rotates about the fixed point O, hence its contribution to H_{11} is constant, $I_3 + m_3 \ell_{C3}^{\ 2}$. Since Link 2 is fixed to ground, Link 4, which is kept parallel to Link 2, does not rotate but translate along a circular trajectory of radius ℓ_1. Therefore, its contribution to H_{11} is $m_4 \ell_1^{\ 2}$, which does not depend on the value of θ_2. Combining these three contributions yields H_{11}, which is therefore constant.



Similarly, H_{22} is the total effective moment of inertial seen by θ_2 while θ_1 is fixed. At this time Link 1 is fixed to ground. The figure depicts the motion of the other three links as θ_2 varies. Links 2 and 4 are fix-point rotations, while Link 3 moves along a circular trajectory of radius ℓ_2 without rotation. These explain each term involved in H_{22}, which does not depend on the value of θ_1.



e) [**Extra credit**]

Recall that kinetic energy can be represented by any set of generalized coordinates as long as they are independent and complete. Endpoint coordinates x_e and y_e, too, are an eligible set of generalized coordinates if the elbow down configuration is restricted. Consider to change the coordinate system from the joint angles to the endpoint coordinates. The Jacobian associated with this coordinate transformation has been obtained in part a),

$$\begin{pmatrix} \dot{x}_e \\ \dot{y}_e \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \quad where\ \mathbf{J} = \begin{pmatrix} -\ell_1 s_1 & \ell_3 s_2 \\ \ell_1 c_1 & -\ell_3 c_2 \end{pmatrix}$$

Applying this to the kinetic energy expression yields

$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{pmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\dot{x}_e & \dot{y}_e\end{pmatrix}(\mathbf{J}^{-1})^T\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\mathbf{J}^{-1}\begin{pmatrix}\dot{x}_e\\ \dot{y}_e\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}\dot{x}_e & \dot{y}_e\end{pmatrix}\begin{bmatrix}H_{11}* & H_{12}*\\ H_{12}* & H_{22}*\end{bmatrix}\begin{pmatrix}\dot{x}_e\\ \dot{y}_e\end{pmatrix}$$

where

$$\mathbf{H}* = \begin{bmatrix}H_{11}* & H_{12}*\\ H_{12}* & H_{22}*\end{bmatrix} = (\mathbf{J}^{-1})^T\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\mathbf{J}^{-1} = \frac{1}{\det\mathbf{J}}\begin{pmatrix}-\ell_3 c_2 & -\ell_1 c_1\\ -\ell_3 s_2 & -\ell_1 s_1\end{pmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\frac{1}{\det\mathbf{J}}\begin{pmatrix}-\ell_3 c_2 & -\ell_3 s_2\\ -\ell_1 c_1 & -\ell_1 s_1\end{pmatrix}$$

$$= \frac{1}{(\det\mathbf{J})^2}\begin{pmatrix}\ell_3 c_2 & \ell_1 c_1\\ \ell_3 s_2 & \ell_1 s_1\end{pmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\begin{pmatrix}\ell_3 c_2 & \ell_3 s_2\\ \ell_1 c_1 & \ell_1 s_1\end{pmatrix}$$

$$= \frac{1}{(\ell_1\ell_3\sin(\theta_2-\theta_1))^2}\begin{bmatrix}\ell_3^2 c_2^2 H_{11} + 2\ell_1\ell_3 c_1 c_2 H_{12} + \ell_1^2 c_1^2 H_{22} & \ell_3^2 s_2 c_2 H_{11} + \ell_1\ell_3\sin(\theta_2+\theta_1)H_{12} + \ell_1^2 s_1 c_1 H_{22}\\ \ell_3^2 s_2 c_2 H_{11} + \ell_1\ell_3\sin(\theta_2+\theta_1)H_{12} + \ell_1^2 s_1 c_1 H_{22} & \ell_3^2 s_2^2 H_{11} + 2\ell_1\ell_3 s_1 s_2 H_{12} + \ell_1^2 s_1^2 H_{22}\end{bmatrix}$$

This new inertia matrix **H*** gives the total effective inertia that the operator feels when moving the endpoint.

Point Allocation

Problem 1 (5 points each) x (8 questions) = 40 points
Problem 2 (10 points each) x (2 questions) = 20 points
Problem 3 (10 points each) x (4 questions) = 40 points

Total 100 points

Class Average = 74.3
ATD = 15.6