Closed-Book. Four sheets of notes are allowed (two from your midterm).
Show how you arrived at your answer.
Do not leave multiple answers. Indicate which one is your correct answer.

Problem 1

A robot is opening a door by holding the door knob shown below. (The robot is not shown.) The distance between the door hinge and the knob is \( R \), and the door knob is free to rotate about its longitudinal axis, i.e. the \( x \) axis of the \( C \)-frame shown in the figure. The process is assumed to be quasi-static and friction-less.

Answer the following questions for performing the task based on Mason’s hybrid position / force control.

(a.) The \( x \) component of translational velocity, \( v_x \), and the \( z \) component of rotational velocity, \( \omega_z \), are coupled to each other due to the geometric constraint created by the door hinge. Consider the \( v_x - \omega_z \) plane shown above in order to describe the direction of admissible motion as well as the constraint space. Let \( v_a \) be a new coordinate axis pointing in the direction of admissible motion space, and \( v_c \) be the one in the direction of the constraint space. Obtain the components of the unit vector along \( v_a \) as well as the ones for \( v_c \).
(b.) Obtain the natural and artificial constraints with respect to the C-frame. Use the new coordinate axes, $v_a$ and $v_c$, as well as the associated force/torque, $f_a$ and $f_c$, for describing the natural and artificial constraints.

(c.) Sketch the block diagram of hybrid position/force control system, and obtain the projection matrices, $P_a$ and $P_c$, and the reference inputs involved in the control system.

Problem 2

Shown below is a schematic of the 2.12 robot arm fixed horizontally to a workbench. Note that the second motor is fixed to the base link and that its torque is transmitted to link 2 through a mass-less belt-pulley mechanism. Joint angles are represented with $\phi_1$ and $\phi_2$, each measured from the $x$ axis. Answer the following questions using the parameters show in the figure.

![Figure 2. Schematic of 2.12 Robot Arm.](image)

(a.) The inertia matrix of the whole system is represented in the following form in terms of joint angles $\phi_1$ and $\phi_2$,

$$H_\phi = \begin{bmatrix} A & D \cos(\phi_2 - \phi_1) \\ D \cos(\phi_2 - \phi_1) & B \end{bmatrix}$$

(b.) Based on the Newton-Euler equations of motion of the individual links, compute the coupling force $f_{1,2}$ acting between Links 1 and 2 when link 2 is accelerated at $\ddot{\phi}_2$ while other velocities and acceleration are: $\dot{\phi}_1 = \ddot{\phi}_1 = \ddot{\phi}_2 = 0$. Also obtain the torque of the first joint needed to cancel the effect of the coupling force induced by the acceleration of link 2, and show that the result agrees with the inertial torque due to the off-diagonal element of the above inertia matrix.

Now the robot arm is placed vertically and the belt-pulley mechanism is modified in order to increase the torque for moving link 2. Namely, the ratio of the motor-side pulley diameter $D_1$ to
the joint-side pulley diameter $D_2$ is now $D_1 : D_2 = 1 : 2$, as shown in Figure 3. For the rest of this
problem, use this ratio. The belt-pulley mechanism is assumed to be mass-less and friction-less.
The displacement of Actuator 2 is denoted $q_2$, and that of Actuator 1 is $q_1$.

![Modified belt-pulley mechanism](image)

Figure 3. Modified belt-pulley mechanism: the ratio of the pulley diameters is now $D_1 : D_2 = 1 : 2$.

c). When Actuator 2 is fixed, Actuator 1 rotates 30 degrees in the positive $q_1$ direction. How
much and in which direction does Link 2 rotate? Based on this observation, obtain the differential
relationship, i.e. the Jacobian matrix, relating joint velocities $\dot{q}_1$, $\dot{q}_2$, to actuator velocities $\dot{q}_1$, $\dot{q}_2$.
d). Obtain the kinetic energy and the potential energy in terms of $q_1$ and $q_2$. Use parameters $A$, $B$, and $D$ obtained in part (a.) for brevity. Also obtain the generalized forces $Q_1$ and $Q_2$ associated
with generalized coordinates $q_1$ and $q_2$, respectively.
e). Obtain the Coriolis and centrifugal torques involved in Lagrange’s equation of motion for
variable $q_1$:

$$
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} \frac{\partial U}{\partial q_1} = Q_1
$$

(2)

Explain why Coriolis terms are generated? (You do not have to obtain the whole equation. Just
compute the Coriolis and centrifugal terms of the first equation alone.)

**Useful Trigonometric Identities**

$$
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \\
\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \\
\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b)
$$
Problem 1

(a.) The geometric constraint created by the door hinge is given by:

\[ v_x = -R \omega_z \]

The unit vector \( \mathbf{a}^* \) in the admissible direction is given by:

\[ \mathbf{a}^* = \left< \frac{1}{1 + R^2}, \frac{-R}{1 + R^2} \right> \]

The orthogonal unit vector \( \mathbf{c}^* \) in the constraint direction is orthogonal to \( \mathbf{a}^* \), and is given by:

\[ \mathbf{c}^* = \left< \frac{R}{1 + R^2}, \frac{1}{1 + R^2} \right> \]

(b.) The table of constraints is as follows:

<table>
<thead>
<tr>
<th></th>
<th>Kinematic</th>
<th>Static</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural</td>
<td>( v_y = 0 ) \n( v_z = 0 ) \n( \omega_y = 0 ) \n( v_c = 0 )</td>
<td>( f_a = 0 ) \n( \tau_x = 0 )</td>
</tr>
<tr>
<td>Artificial</td>
<td>( v_a = \text{constant} ) \n( \omega_x = 0 )</td>
<td>( f_y = 0 ) \n( f_z = 0 ) \n( \tau_y = 0 ) \n( f_c = 0 )</td>
</tr>
</tbody>
</table>

(c.) The figure below shows the structure of the hybrid position/force controller with the reference inputs:

\[ \mathbf{p}_{\text{ref}} = \begin{bmatrix} Vt & 0 & 0 & 0 & 0 \frac{-Vt}{R} \end{bmatrix} \]

and

\[ \mathbf{f}_{\text{ref}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

and projection matrices (for coordinates \([v_x \ v_y \ v_z \ \omega_x \ \omega_y \ \omega_z]\)).

\[ \mathbf{P}_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathbf{P}_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Other acceptable answers are possible expressing the control in terms of \( \mathbf{a}^* \) and \( \mathbf{c}^* \).
Figure 1: Hybrid Position/Force Controller.
Problem 2

(a.) Kinetic Energy:

\[
T = \frac{1}{2} \dot{\phi}_1^2 + \frac{1}{2} m_1|v_{c1}|^2 + \frac{1}{2} \dot{\phi}_2^2 + \frac{1}{2} m_1|v_{c2}|^2
\]

\[
|v_{c1}|^2 = l_{c1}^2 \dot{\phi}_1^2
\]

\[
v_{c2} = \frac{d}{dt} \begin{pmatrix} l_1c_1 + l_2c_2 \\ l_1s_1 + l_2s_2 \end{pmatrix} = \begin{pmatrix} -l_1s_1 - l_2s_2 \\ l_1c_1 \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}
\]

\[
|v_{c2}|^2 = v_{c2}^Tv_{c2} = (\dot{\phi}_1 \ \dot{\phi}_2) \begin{pmatrix} l_1^2 & l_1l_2 \cos(\phi_2 - \phi_1) \\ l_1l_2 \cos(\phi_2 - \phi_1) & l_2^2 \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}
\]

\[
T = \frac{1}{2} I_1 \dot{\phi}_1^2 + \frac{1}{2} m_1 l_{c1}^2 \dot{\phi}_1^2 + \frac{1}{2} I_2 \dot{\phi}_2^2 + \frac{1}{2} m_2 (l_1^2 \dot{\phi}_1^2 + 2 l_1 l_2 \cos(\phi_2 - \phi_1) \dot{\phi}_1 \dot{\phi}_2 + l_2^2 \dot{\phi}_2^2)
\]

\[
T = \frac{1}{2} \begin{pmatrix} I_1 + m_1 l_{c1}^2 + m_2 l_1^2 \\ m_2 l_1 l_2 \cos(\phi_2 - \phi_1) \end{pmatrix} \dot{\phi}_1^2 + m_2 l_1 l_2 \cos(\phi_2 - \phi_1) \dot{\phi}_1 \dot{\phi}_2 + \frac{1}{2} \begin{pmatrix} I_2 + m_2 l_2^2 \end{pmatrix} \dot{\phi}_2^2
\]

\[
T = \frac{1}{2} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} \begin{pmatrix} I_1 + m_1 l_{c1}^2 + m_2 l_1^2 & m_2 l_1 l_2 \cos(\phi_2 - \phi_1) \\ m_2 l_1 l_2 \cos(\phi_2 - \phi_1) & I_2 + m_2 l_2^2 \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}
\]

therefore:

\[
h_{\phi} = \begin{pmatrix} i_1 + m_1 l_{c1}^2 + m_2 l_1^2 & m_2 l_1 l_2 \cos(\phi_2 - \phi_1) \\ m_2 l_1 l_2 \cos(\phi_2 - \phi_1) & i_2 + m_2 l_2^2 \end{pmatrix}
\]

and

\[
A = I_1 + m_1 l_{c1}^2 + m_2 l_1^2
\]

\[
B = I_2 + m_2 l_2^2
\]

\[
D = m_2 l_1 l_2
\]

\[
H_{\phi,11} = a
\]

is the moment of inertia seen by the first joint \( \phi_1 \) when \( \phi_2 \) is immobilized. link 2 does not rotation with \( \phi_2 \) fixed. therefore:

1. the moment of inertia of link 2, \( i_2 \), does not contribute to \( H_{\phi,11} \), and

2. link 2 translates as \( \phi_1 \) changes. the trajectory of the mass centroid of link 2 is a circle of radius \( l_1 \), regardless of the arm configuration. therefore, the contribution of link 2 to \( H_{\phi,11} \)

is constant, i.e., configuration invariant.

(b.) From the free-body diagram of link 2 shown above

\[
m_2 \ddot{v}_{c2} = f_{1,2}
\]
since $\ddot{\phi}_1 = \dot{\phi}_1 = \dot{\phi}_2 = 0$

\[ \dot{v}_{c2} = l_{c2} \begin{pmatrix} -\sin \phi_2 \\ \cos \phi_2 \end{pmatrix} \ddot{\phi}_2 \]

therefore:

\[ f_{1,2} = m_2 l_{c2} \begin{pmatrix} -\sin \phi_2 \\ \cos \phi_2 \end{pmatrix} \ddot{\phi}_2 \]

The torque, $\tau'_1$, needed for canceling the effect of the coupling force $f_{1,2}$ on link 1 is:

\[ \tau'_1 + r_{0,1} \times (-f_{1,2}) = 0 \]

\[ \tau'_1 = r_{0,1} \times f_{1,2} = l_1 \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} \times m_2 l_{c2} \begin{pmatrix} -s_2 \\ c_2 \end{pmatrix} \ddot{\phi}_2 \]

\[ \tau'_1 = m_2 l_1 l_{c2} \ddot{\phi}_2 (c_1 c_2 + s_1 s_2) = m_2 l_1 l_{c2} \ddot{\phi}_2 \cos(\phi_2 - \phi_1) \ddot{\phi}_2 \]

\[ \tau'_1 = d \cos(\phi_2 - \phi_1) \ddot{\phi}_2 \]

this is the same as the off-diagonal element of the inertia matrix, i.e., $H_{\phi_{12}}$, which represents the coupling (interactive) inertia between the two joints.
(c.) Figure C-1 aboves shows the motion of links 1 and 2 when actuator 1 rotates $30^\circ$ and actuator 2 is fixed; $\Delta q_1 = 30^\circ$, $\Delta q_2 = 0^\circ$. When sitting on link 1 and viewing the motion of the motor side pulley, we find that it rotates $-30^\circ$ (in the clockwise direction). This pulls the belt $\frac{D_1}{D_2} \Delta q_1$ in the direction shown in Figure C-2. As a result, the pulley on the second link side rotates $\frac{D_1}{D_2} \Delta q_1$ relative to link 1. As shown in Figure C-1, this rotates link 2, which is attached to the pulley of diameter $D_2$, in the negative (clockwise) direction. Since link 1 itself rotates $\Delta q_1$, the resultant rotation of link 2 viewed from the $x$-axis is given by:

$$\Delta \phi_2 = \Delta q_1 - \frac{D_2}{D_1} \Delta q_1 = (1 - \frac{D_1}{D_2}) \Delta q_1 = (1 - \frac{1}{2}) 30^\circ = 15^\circ$$

Therefore, link 2 rotates $15^\circ$ in the positive (counter-clockwise) direction, as shown in Figure C-1. Note that the coefficient $(1 - \frac{D_1}{D_2})$ gives the partial derivatives of $\phi_2$ to $q_1$:

$$\frac{\partial \phi_2}{\partial q_1} = 1 - \frac{D_1}{D_2}$$

Since the above $\Delta \phi_2$ is generated by $\Delta q_1$ alone, and $\Delta q_2$ is kept zero. As the second actuator moves ($\Delta q_2 \neq 0$), link 2 rotates $\frac{D_1}{D_2} \Delta q_2$ in the same direction as the actuator. Therefore, the total displacement of link 2 is:

$$d \phi_2 = \frac{\partial \phi_2}{\partial q_1} dq_1 + \frac{\partial \phi_2}{\partial q_2} dq_2 = (1 - \frac{D_1}{D_2}) dq_1 + \frac{D_1}{D_2} dq_2$$

Or, in velocity form:

$$\dot{\phi}_2 = (1 - \frac{D_1}{D_2}) \dot{q}_1 + \frac{D_1}{D_2} \dot{q}_2$$

Since $\phi_1 = q_1$ ($\dot{\phi}_1 = \dot{q}_1$),

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 1 - \frac{D_1}{D_2} & 0 \\ \frac{D_1}{D_2} & \frac{D_1}{D_2} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$
hence:

\[ J = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \]  

(1)

check: if \( D_1 = D_2 \), then \( J \) reduces to \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \dot{\phi}_1 = \dot{q}_1, \dot{\phi}_2 = \dot{q}_2 \). this agrees with the case of our “standard” 2.12 robot arm.

(d.) from part (a):

\[
T = \frac{1}{2} \begin{pmatrix} \dot{\phi}_1 & \dot{\phi}_2 \end{pmatrix} \begin{pmatrix} A & D \cos(\phi_2 - \phi_1) \\ D \cos(\phi_2 - \phi_1) & B \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}
\]

Denoting \( C_{2-1} = \cos(\phi_2 - \phi_1) \) and substituting \( \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \) into the kinetic energy yields:

\[
T = \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} J' \begin{pmatrix} A & D C_{2-1} \\ D C_{2-1} & B \end{pmatrix} J \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
\]

\[
T = \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \left( A + D C_{2-1} + \frac{1}{4} B \right) \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
\]

We can define \( H_q \) as follows:

\[
H_q = \begin{pmatrix} A + D C_{2-1} + \frac{1}{4} B & \frac{1}{2} D C_{2-1} + \frac{1}{4} B \\ \frac{1}{2} D C_{2-1} + \frac{1}{4} B & \frac{1}{4} B \end{pmatrix}
\]

Next, define the potential energy:

\[
U = m_1 g l_1 \sin \phi_1 + m_2 g (l_1 \sin \phi_1 + l_2 \sin \phi_2)
\]

From part (c), \( \phi_1 = q_1, \phi_2 = \frac{1}{2} q_1 + \frac{1}{2} q_2 \)

\[
U = m_1 g l_1 \sin q_1 + m_2 g (l_1 \sin q_1 + l_2 \sin \frac{1}{2} (q_1 + q_2))
\]

Next, generalized forces: consider the virtual work (see figure below)

\[
\delta \text{Work} = \tau_1 \delta \phi_1 + \tau_2 \delta \phi_2
\]

\[
= \tau_1 \delta q_1 + \tau_2 \delta \left( \frac{1}{2} q_1 + \frac{1}{2} q_2 \right)
\]

\[
= (\tau_1 + \tau_2) \delta q_1 + \frac{1}{2} \tau_2 \delta q_1
\]
\[ Q_1 = \tau_1 + \frac{1}{2} \tau_2 \]
\[ Q_2 = \frac{1}{2} \tau_2 \]

(e.) The Coriolis and centripetal terms come from the configuration dependent nature of the inertia matrix. In the above \( H_q \), only \( C_{2-1} = \cos(\phi_2 - \phi_1) \) is configuration dependent. Rewriting the kinetic energy in two parts:

\[
T = \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} DC_{2-1} & \frac{1}{2} DC_{2-1} \\ \frac{1}{2} DC_{2-1} & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} A + \frac{1}{4} B & \frac{1}{2} B \\ \frac{1}{2} B & \frac{1}{4} B \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
\]

we see that the first term is configuration dependent, and hence contains the Coriolis and centripetal terms, and the second term is configuration-invariant (no Coriolis and centripetal terms).

Let us define a new kinetic energy \( T' \), only reflecting the first term:

\[
T' = \frac{1}{2} (\dot{q}_1^2 + \dot{q}_1 \dot{q}_2) DC_{2-1}
\]

Next, let’s compute the terms in the Lagrange’s equation

\[
\frac{d}{dt} \frac{\partial T'}{\partial \dot{q}_1} = \frac{d}{dt} (\dot{q}_1 + \frac{1}{2} \dot{q}_2) D \cos \frac{1}{2} (q_2 - q_1)
\]

\[
= (\dot{q}_1 + \frac{1}{2} \dot{q}_2) D \frac{d}{dt} \cos \frac{1}{2} (q_2 - q_1) + (\ddot{q}_1 + \frac{1}{2} \ddot{q}_2) D \cos \frac{1}{2} (q_1 - q_2)
\]

The second term has no Coriolis or centripetal terms, so we can ignore it. Next let us evaluate the derivative in the first term:

\[
D(\dot{q}_1 + \frac{1}{2} \dot{q}_2) \left\{ -\sin \left( \frac{1}{2} (q_2 - q_1) \right) \left( -\frac{1}{2} \ddot{q}_1 \right) - \left( \sin \left( \frac{1}{2} (q_2 - q_1) \right) \right) \right\}
\]

\[
\frac{1}{2} D(\dot{q}_1 + \frac{1}{2} \dot{q}_2)(\dot{q}_1 - \dot{q}_2) \sin \frac{1}{2} (q_2 - q_1)
\]

(2)

Now the other term we need for applying Lagrange’s equation:

\[
-\frac{\partial T'}{\partial q_1} = -\frac{1}{4} D(\ddot{q}_1^2 + \dot{q}_1 \dot{q}_2) \sin \frac{1}{2} (q_2 - q_1)
\]

(3)

Combining Equations 2 and 3 we can obtain all the Coriolis and centripetal terms,

\[
\frac{1}{4} D \sin \frac{1}{2} (q_2 - q_1) \cdot (\dot{q}_1^2 - \dot{q}_2^2 - 2 \dot{q}_1 \dot{q}_2)
\]

The Coriolis term \(-\frac{1}{2} D \sin \frac{1}{2} (q_2 - q_1) \cdot \dot{q}_1 \dot{q}_2\) now appears since Link 2 rotates as Link 1 rotates.