AN INTRODUCTION TO LAGRANGE EQUATIONS

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1.0   INTRODUCTION

This paper is intended as a minimal introduction to the application of Lagrange equations to the task of finding the equations of motion of a system of rigid bodies. A much more thorough and rigorous treatment is given in the text "Fundamentals of Applied Dynamics" by Prof. James H. Williams, Jr., published in 1996 by John Wiley and Sons. In particular the treatment given here to the evaluation of non-conservative external forces is simple and pragmatic. It will suffice for most of the applications at the level of an undergraduate course in engineering dynamics. Deeper understanding requires more work and the reader is referred to the Williams text. In writing this summary I have borrowed liberally from the Williams text and also from a previous subject handout, prepared by Prof. Sanjay Sarma of the Mechanical Engineering Department of MIT.

2.0   ENERGY METHODS IN STATICS

Principle of virtual work: Energy methods were in use to solve statics problems before discovering them for dynamics. One example is the principal of virtual work. If a structure such as the one shown in the diagram below is in static equilibrium then from the principle of virtual work we can deduce the static equilibrium equation by computing the virtual work done during an infinitesimal deflection. In this case the uniform bar, shown in Figure 1, rotates about the hinge at A.

It is acted upon by its weight, Mg, applied at its center of gravity at B, a spring force, F_s, applied at its point of attachment at C, and by the workless constraint reaction force, R_y, applied at A.   Assume the static equilibrium position is horizontal and assume a small virtual angular displacement, $\delta\theta$. The principal of virtual work tells us that the work done by the all of the forces in the course of the virtual displacement is zero. Putting this into equation form yields directly the relationship between the applied external gravitational force and the internal spring force as shown below.
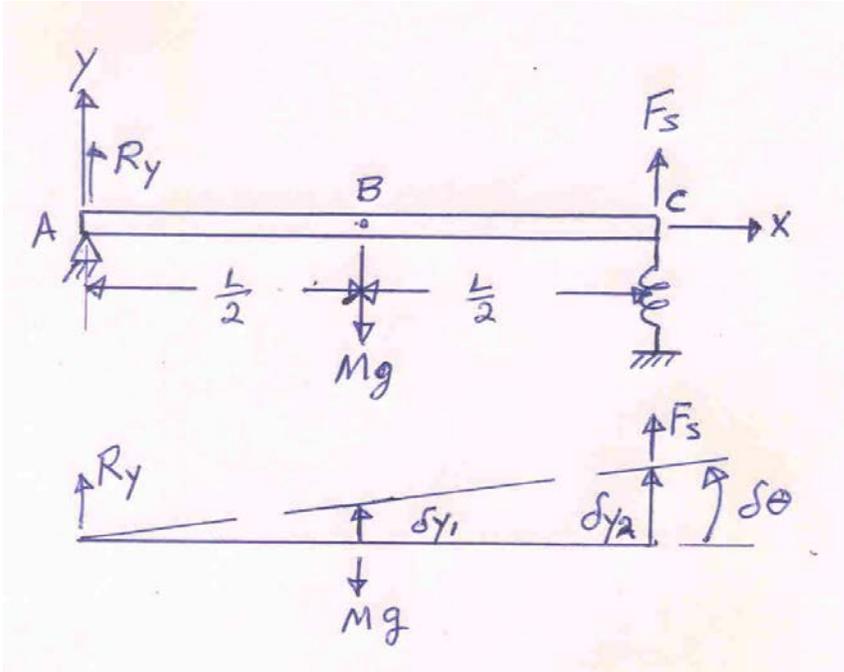
Figure 1. Hinged bar with a small virtual displacement

$$\delta W = [R_y \bullet 0 - Mg\delta y_1 + F_s\delta y_2] = 0$$

Noting that $\delta y_1 = \dfrac{L}{2}\delta\theta$ **and** $\delta y_2 = L\delta\theta$, **then** $[-Mg\dfrac{L}{2} + F_sL]\delta\theta = 0$

Since $\delta\theta \neq 0,$ **then the part inside the brackets must equal zero.**

$$[-Mg\dfrac{L}{2} + F_sL] = 0,$$ **the condition for static equilibrium.**

(2.1)

Solving for the unknown spring force yields

$$F_{spring} = \dfrac{Mg}{2}$$

An important feature to note is that R_y, the reaction force at the hinge does not appear in the final expression, because it does no work during the virtual displacement. This is a benefit that carries over to dynamical systems when the concept of virtual work is extended to application of Lagrange equations to problems in dynamics. The equations of motion can be found without having to solve for workless constraint forces.

Principle of minimum potential energy: Another related concept came into use for the analysis of elastic structures for which it was possible to evaluate the total potential energy of the system, including strain energy stored in structural elements and strain energy due to gravitational loads. This can be illustrated by considering a simple weight hanging on a spring, as shown in Figure 2. This is a system in which only one coordinate, x is needed to describe the vertical deflection of the system.
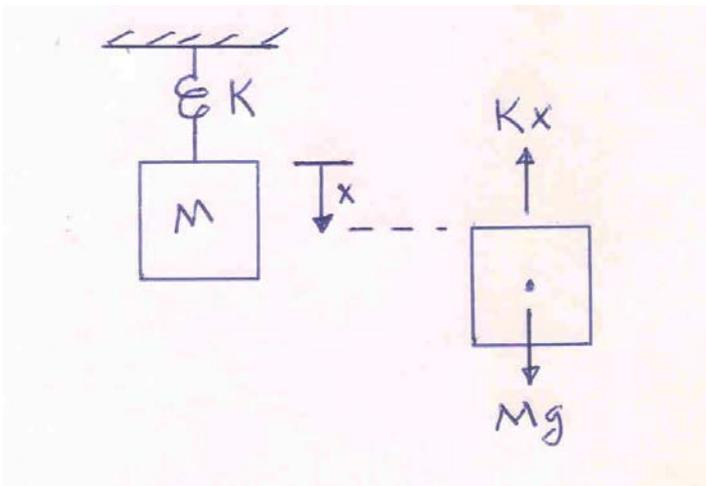
Figure 2. Mass spring system.

As shown in the diagram, the origin is taken to be at the position where the spring force is zero. The deflection, x, is positive downwards. The total potential energy of the system for a positive deflection of the mass may be expressed as:

$$V = -\int F \cdot dx = \frac{1}{2}Kx^2 - Mgx \qquad (2.2)$$

The principle of minimum potential energy states that "Of all the states which satisfy the boundary conditions(constraints), the correct state is that which makes the total potential energy minimum." Mathematically this can be expressed as:

$$\frac{\partial V}{\partial x} = kx - Mg = 0 \qquad (2.3)$$
$$\Rightarrow x = Mg / k$$

Thus, from knowing the total potential energy of the system, and by applying this principle, one may directly obtain the static equilibrium equation as well as the deflection, x, as a function of the applied load, Mg.

There is a very useful conceptual link between the principle of virtual work and the principle of minimum potential energy.   If we imagine a small virtual deflection of the system about its static equilibrium position then we can modify the equations obtained above to define the variation in the potential energy in terms of the virtual work, which results from a small virtual displacement, δx , as shown below. The two expressions on the left hand side represent the variation in the potential energy whereas the expression on the right hand side shows the virtual work. Both are equal to zero because this is a small variation about a static equilibrium position for a system with no non-conservative forces acting on it.

$$\delta V = \frac{\partial V}{\partial x} \delta x = (kx - Mg)\delta x = 0 \qquad (2.4)$$

In fact the term $\frac{\partial V}{\partial x}\delta x$ is in the form of the potential energy term which appears in the Lagrange equations defined in the remainder of this document.

3.0   AN INTRODUCTION TO ENERGY METHODS IN DYNAMICS

Newton published the *Principia* in 1687, in which he laid down his three laws of motion. 101 years later, in 1788, Lagrange developed a way to derive the equations of motion by beginning with work and energy rather than force and momentum. One way to think about the leap from the statics problems described above and a dynamics problem is to consider what textbooks refer to as the work energy theorem, which may be stated as:

$$(T_2 - T_1) + (V_2 - V_1) = W_{1 \rightarrow 2}^{nc} \qquad (3.1)$$

This states that the change in kinetic energy, T, and potential energy, V, as a dynamic system moves from state 1 to 2 is equal to the non-conservative work that is done by the external non-conservative forces applied to the system, as it moves from position 1 to 2. At all times in the movement of this system, the above statement must be true, if the system is in dynamic equilibrium. Another way of saying this is to say that the system would satisfy the equations of motion at all times.

Now imagine that at an instant in time, there is an infinitesimal variation in the position of the system from its natural dynamic equilibrium position. Just as is the case for the principal of virtual work for systems in static equilibrium, the net variation in the kinetic and potential energy and the work of the non-conservative forces would be zero. The variation in energy or work in each term is not zero, but the net sum is zero. Stated in equation form:

$$\delta T + \delta V - \delta W^{nc} = 0$$

Or if one prefers: $\qquad (3.2)$

$$\delta T + \delta V = \delta W^{nc}$$

The second expression says that the change in the sum of the kinetic and potential energies of the system must equal the work done by the external non-conservative forces. Lagrange discovered a way to express this for multiple degree of freedom systems. Before jumping directly to the equations, it is essential to carefully explain how one determines the correct number of necessary coordinates and degrees of freedom for any dynamical system that is to be modeled using the Lagrangian approach. There are even some dynamical systems for which Lagrange equations may not be used. One needs to know how to identify them.

4.0 GENERALIZED COORDINATES AND DEGREES OF FREEDOM

Generalized coordinates: The generalized coordinates of a mechanical system are the minimum group of parameters which can completely and unambiguously define the configuration of that system. They are called generalized because they are not restricted to being Cartesian coordinates and are not even required to be measured from an inertial reference frame. However, they are used to express the kinetic and potential energies of the rigid bodies which make up the dynamical system. The kinetic and potential energy must be computed with respect to an inertial reference frame. Therefore the generalized coordinates must be able to express the velocity and displacement of rigid bodies with respect to an inertial frame of reference.

Holonomic systems: A *holonomic* system is one in which the number of independent generalized coordinates required to describe the motion of the system equals the number of degrees of freedom.   We study only holonomic systems in this class. An example of a non-holonomic system is given in lecture and others may be found in the text "Fundamentals of Applied Dynamics" by Prof. James H. Williams, Jr.

Degrees of freedom: The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. Said another way, the number of degrees of freedom is the same as the number of independent directions in which a system can move from any given initial configuration. Consider a system consisting of N rigid bodies in 2D space. Each rigid body has 3 degrees of freedom: two translational and one rotational. The N-body system has $3N$ degrees of freedom. Now let's say that there are k kinematic constraints. Then the system has $d = 3N - k$ degrees of freedom. For example a single uniform stick confined to a vertical, x-y plane has 3 degrees of freedom, x, y, and rotation, ϑ about the z axis. It has no additional constraints. If we connect one end of the stick to a pivot, it now has two constraints, one in each of the x and y directions. It is still free to rotate. Therefore *d=3-2=1 dof.* The motion of the pinned stick may be completely specified by one coordinate, the angle of rotation.

Completeness and Independence of generalized coordinates: In choosing which coordinates to use one must obey two requirements.
1. A set of generalized coordinates is *complete*, if it is capable of locating all parts of the system at all times.
2. A set of generalized coordinates is *independent* if, when all but one of the generalized coordinates are fixed, there remains a continuous range of values for that one coordinate.

There you have it. To find the equations of motion of a dynamical system using Lagrange equations, one must first determine the number of degrees of freedom, 'd', and then choose a set of 'd' generalized coordinates, which make up a complete and independent set.

5.0   THE LAGRANGE EQUATIONS

For a j-dof (degree-of-freedom) system with generalized coordinates q_j, it is possible to formulate the Lagrangian $L = T - V$ where T is the kinetic energy and V is the potential energy. The Lagrangian is a function of generalized coordinates q_j and generalized velocities $\dot{q}_j$:

$$L = L\left(q_1, \ldots q_j \ldots q_d,\ \dot{q}_1, \ldots \dot{q}_j \ldots \dot{q}_d\right) \tag{4.1}$$

where d is the number of degrees of freedom. The Lagrange Equations are then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{4.2}$$

where Q_j are the external generalized forces. Since j goes from 1 to d, Lagrange gives us d equations of motion the same number as the degrees of freedom for the system.

The left hand side of Equation 4.2 is a function of only T *and* V, the potential energy and kinetic energy and is straightforward to evaluate once the analyst has found T & V. The right hand side is where most difficulties arise. The right hand side requires that one take the real non-conservative forces that one would specify on a free body diagram and re-express them in terms of the component of the real forces that act in the direction of each generalized deflection. This is the process of finding the generalized forces.

6.0   FINDING THE GENERALIZED FORCES

Let's consider first just one external real force, applied at some point on the body. An example, borrowed from Williams is shown in the figure below. It is a hinged rigid link, which is acted upon by an external force, F(t) which always points down. This system has one degree of freedom and therefore the angle, θ, that the link makes with the vertical has been specified as a complete and independent set of generalized coordinates.

For this example,
$$q_j = q_1 = \theta$$
$\delta q_1 = \delta\theta$, a small virtual variation in θ

and $Q_1\delta\theta = \boldsymbol{F}\cdot\delta\boldsymbol{r} =$ the work done by the external force F during the variation $\delta\theta$.

The magnitude of the generalized force, $\boldsymbol{Q}_1$, associated with the virtual deflection of the system, $\delta\theta$, may be obtained by finding the virtual work done by the real force F(t) in the course of the virtual displacement.

In equation form, the work done is the dot product of the applied external force, F(t), and the virtual deflection in the direction of the generalized coordinate which is going through the variation $\delta\theta$. In this case the generalized coordinate is θ, and the deflection component in the direction of F(t) is $l\sin(\theta)\delta\theta$, as shown in part c of the figure below. The virtual work done during this small variation is given by

$$\delta W^{nc} = Q_1 \delta\theta = F(t)l\sin(\theta)\delta\theta$$

Therefore (4.3)

$$Q_1 = F(t)l\sin(\theta)$$

In this example, the actual displacement at the location of and in the direction of the force F(t) is given by $l\sin(\theta)\delta\theta$, as shown in the diagram above.

(b) Rigid link loaded by vertical force, showing geometry of admissible variation.

Undisplaced position of link

Work-producing variation in direction of $F(t)$

$F(t)$

$F(t)$

(a) Rigid link loaded by force that remains in negative Y direction.

Work-producing variation in direction of $F(t)$

(c) Expanded view of admissible variation at point of force application.

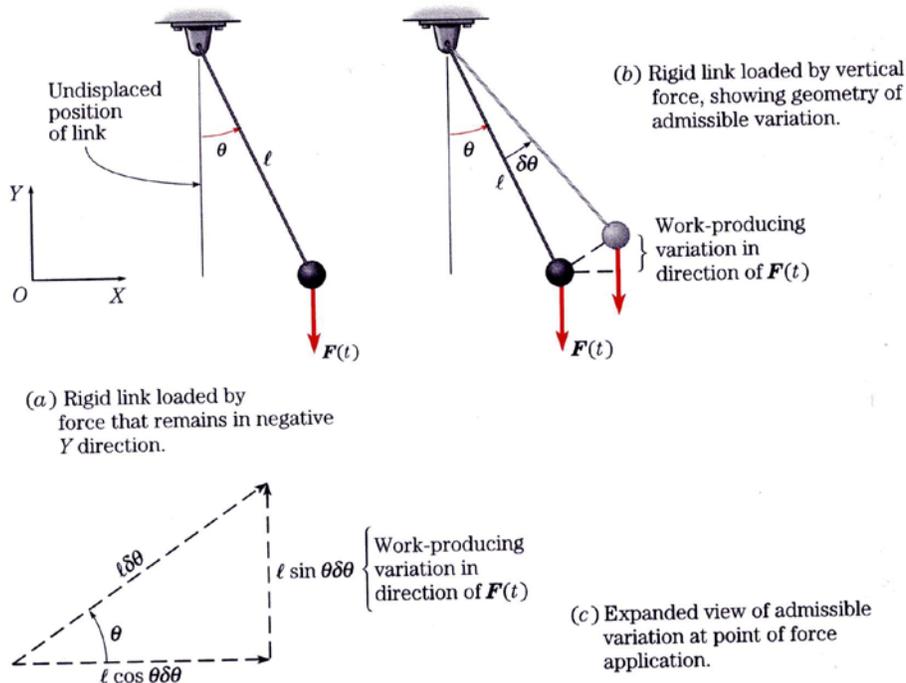$\ell\sin\theta\delta\theta$

$\ell\cos\theta\delta\theta$

FIGURE 5-20   Rigid link loaded by vertical force, and associated geometry for admissible variation.

For many problems the dot product of the external force and the deflection component in the direction of each generalized displacement may be deduced by common sense as was done above. Several examples are given in the chapter by Williams and more will be done in lecture.

In the above example the system had only one degree of freedom and therefore there was only one virtual displacement of the single coordinate θ. In general there will be many coordinates and many applied external forces. We need to broaden the approach to account for this.

At every point of application of a real external force, $\mathbf{F_i}$, to the system work will be done if that point moves. Some external forces like those produced by gravity or lossless springs are usually included in the potential energy expression. We will assume so here. The ones remaining are called non-conservative forces because they may add or extract energy from the system. The technique used to evaluate these non-conservative generalized forces is to assume the system undergoes a small virtual deflection for which we compute the increment or variation in work that results, as expressed in the following

equation, which says that the total variation in the work of the non-conservative forces is given by the sum of the dot products of the external forces with the total virtual deflection at the location of each force, due to the combined virtual deflections of each coordinate.

$$\delta W^{nc} = \sum_{i=1}^{N} F_i \cdot \delta r_i (\delta q_1, \delta q_2 \dots \delta q_d) = \sum_{j=1}^{d} Q_j \delta q_j \tag{4.4}$$

The term on the right of equation 4.4 says that the same total variation in the work done by the non-conservative forces must equal the sum of the generalized forces, Q_j, multiplied by the virtual displacement δq_j associated with that generalized force. In this course we will do problems for which we can separate out the individual terms by applying geometry and simple vectors. We will in general consider the virtual work done by considering the variation of one coordinate at a time and from that deduce the generalized force associated with that motion.

A more rigorous method exists which is described in a brief appendix at the end of this paper.

In summary, some key points to remember regarding generalized forces are:

- Forces of constraint which do no work can be ignored because they will always be perpendicular to allowable motions, also known as admissible motions. Examples include the internal forces in a rigid body, the forces of reaction in friction-less sliding, and so on.
- Forces which derive from a potential function like gravity or a spring can be considered in potential energy, V. They too can be ignored when computing generalized forces.
- Forces which are neither included in the computation of potential energy nor are workless forces of constraint must be expressed as non-conservative generalized forces in the Lagrange formulation. Examples include dissipative forces from dashpots, external excitations. If one mistakenly accounts for a potential force as a non-conservative force, it will work out just fine in the end, as long as you don' also account for it in the potential energy expression.

7.0   USING LAGRANGE'S EQUATIONS

Now we can return to the Lagrange equations as given in Equation 4.2.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{4.5}$$

Mathematicians like the elegance of this expression, but it obscures some of the physics. Lagrange was proud of the fact that his 1788 treatise did not contain a single diagram.

Let's reconnect it to the physics. Recall that the Lagrangian, L is defined as $L=T-V$. Substitution in the above equation yields:

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial\dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}=\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_j}\right)-\frac{d}{dt}\left(\frac{\partial(V)}{\partial\dot{q}_j}\right)-\frac{\partial(T)}{\partial q_j}+\frac{\partial(V)}{\partial q_j}=Q_j \qquad \text{a.}$$

Noting that $\dfrac{d}{dt}\left(\dfrac{\partial(V)}{\partial\dot{q}_j}\right)=0$ for common mechanical systems,

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_j}\right)-\frac{\partial(T)}{\partial q_j}+\frac{\partial(V)}{\partial q_j}=Q_j \qquad \text{b.}$$

Multiplying by $\delta q_j \Rightarrow$ \hfill (4.6)

$$\left[\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_j}\right)-\frac{\partial(T)}{\partial q_j}\right]\delta q_j+\frac{\partial(V)}{\partial q_j}\delta q_j=Q_j\delta q_j \qquad \text{c.}$$

This is of the form

$$\delta T+\delta V=\delta W^{nc} \qquad \text{d.}$$

In common mechanical systems, potential energy accounts for gravitational potential energy or the potential energy stored in springs. The potential energy for such systems is never a function of velocity, and hence the term $\dfrac{d}{dt}\left(\dfrac{\partial(V)}{\partial\dot{q}_j}\right)$ vanishes in the above equations. For simple mechanical systems the term $-\dfrac{\partial(T)}{\partial q_j}$ is often zero. The form of Lagrange equations expressed above in terms of T and V and not L is therefore easier to use and requires less wasteful bookkeeping.

$$\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_j}\right)-\frac{\partial(T)}{\partial q_j}+\frac{\partial(V)}{\partial q_j}=Q_j$$

Valid for pure mechanical systems only, \hfill (4.7)

no electric or magnetic fields.

It may be used for all mechanical problems of the kind done in this course. When one should be cautious with this simplification is when there are potential forces involving charged particles and/or electric currents and magnetic fields. Then the assumption that $\dfrac{d}{dt}\left(\dfrac{\partial(V)}{\partial\dot{q}_j}\right)$ vanishes is not valid.

The steps in computing the equations of motion using Lagrange's method are below.

Start with the LHS of Equation 4.2 or the more direct 4.7:

1. Identify the generalized coordinates. Make sure that you have just as many as there are degrees-of-freedom and confirm that the system is holonomic.

2. Compute the kinetic energy T as a function of q_j's and $\dot{q}_j$'s.

3. Compute the potential energy V as a function of q_j's. Clearly designate the forces which you will call out as potential and forces and those which you will call out as non-conservative external forces.

4. Compute separately $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_j}\right)$, $-\dfrac{\partial T}{\partial q_j}$ and $\dfrac{\partial V}{\partial q_j}$ for each generalized coordinate and from the sum of the three pieces you will have the LHS of the equation of motion associated with each generalized coordinate j, as given in equation 4.6b. The LHS by itself represents the equations of motion of the system without the action of external non-conservative forces. It is useful by itself in understanding the system.

Now the RHS of Equation 4.2 and 4.7:

1. Identify all non-conservative forces. They may be many or few and have nothing to do with the number of degrees of freedom in the problem.

2. Number them as $i = 1, 2, ... n$. Call the forces $\mathbf{F}_1, \mathbf{F}_2, ... \mathbf{F}_n$.

3. For each j, compute the virtual work associated with the virtual deflection of each generalized coordinate. $\boldsymbol{Q}_j \delta \boldsymbol{q}_j = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \left(\delta \boldsymbol{q}_j \right)$ where each $\delta \boldsymbol{r}_i$ is a function of but not necessarily equal to $\delta \boldsymbol{q}_j$ as was true in the first example which considered the motion of the hinged link. The RHS of this expression will be of the form $[?]\delta \boldsymbol{q}_j$, where the term in [ ] will involve the real forces, F_i, and geometry. This term [ ] = Q_j is the generalized force you are seeking to put into the RHS of equation 4.7 or equation 4.2.

4. For more difficult problems employ the explicit computation prescribed by the equation: $Q_j \delta q_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\dfrac{\partial r_i}{\partial q_j} \right) \delta q_j$ which is described in the appendix.

5. Now equate the LHS to RHS of equation 4.7 for each j. You will finish with d equations of motion, one for each degree of freedom.

The final step is to put this to practice doing lots of examples. Examples are to be found in lecture, in the course problem sets and in the text by Williams.

8.0   ADDVANTAGES AND DISADVANTAGES OF LAGRANGE

Advantages:
1. The Lagrange Approach automatically yields as many equations as there are degrees of freedom.

2. The Lagrange equations naturally use the generalized coordinates of the system. By contrast, Newton's Equations are essentially Cartesian. You end up having to convert everything into Cartesian components of acceleration and Cartesian components of forces to use Newton's Equation. Lagrange bypasses that conversion.

3. The Lagrange approach naturally eliminates forces of constraint. You could do the same with the direct (Newtonian) approach, but your ability to minimize the number of variables depends very much on your skill; Lagrange takes care of it for you automatically because the generalized forces only include force components in directions of admissible motion.

Disadvantages:

1. If you make mistakes in formulation of T, V and W^{nc} you may never notice. The physics of forces and free body diagrams is easily overlooked. One should always evaluate each of the equations obtained by the Lagrange method and question whether or not it makes physical sense. Do the expected accelerations occur? For example for systems which rotate are there the expected centripetal and/or coriolis accelerations.

2. Computation of the generalized forces can be as much or more work in the Lagrangian method as in the direct method.

9.0   APPENDIX: A KINEMATICS APPROACH TO COMPUTING GENERALIZED FORCES

In order to use the Lagrange method one must find the Generalized forces, Q_j. What one must identify first are the external forces, $\mathbf{F_i}$, and then must find a way to express the arbitrary virtual displacement δr_i in terms of the variations of the generalized displacements, δq_j.

The generalized force Q_j is defined below:

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial r_i}{\partial q_j} \right) \qquad\qquad (9.1)$$

where $\boldsymbol{F_i}$ is the vector force at point i and $\mathbf{r}_i$ is the position vector to point i, measured with respect to an inertial frame of reference. The index j corresponds to each of the generalized coordinates, which must be evaluated one at a time. The summation over i accounts for the fact that there may be more than one real force, $\boldsymbol{F_i}$, which does work when the virtual displacement δq_j takes place. It is the total of all these contributions that makes up the generalized force Q_j. The total virtual work associated with the virtual displacement δq_j is the sum of the virtual work done by each of the real forces $\boldsymbol{F_i}$ as a result of the virtual displacement δq_j.

$$\delta W_j^{nc} = \boldsymbol{Q}_j \delta \boldsymbol{q}_j = \sum_{i=1}^{N} \mathbf{F}_i \bullet \left(\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{q}_j} \right) \delta \boldsymbol{q}_j \tag{9.2}$$

It is from this equation that one deduces the generalized force that is needed in each equation of motion yielded by application of equation 4.7.

At the location of F_i a small variation of δq_j results in a deflection at i in the j direction, $\delta \mathbf{r}_{ij}$, given by:

$$\delta \mathbf{r}_{ij} = \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{9.3}$$

$\delta \mathbf{r}_{ij}$ is never in the direction of a kinematic constraint, because no virtual displacement is allowed in the direction of a constraint. In general, for a d-degree-of-freedom system with generalized coordinates $q_1, ..., q_d$, the total admissible motion at a point i with position vector $\mathbf{r}_i$ is given by a summation over the contributions from the small virtual deflection of every generalized coordinate that affects the motion at position i, as specified below.

$$\delta \mathbf{r}_i = \sum_{j=1}^{d} \delta \mathbf{r}_{i,j} = \sum_{j=1}^{d} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{9.4}$$